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## Analysis of correlation functions in Toda theory and AGT-W relation for $SU(3)$ quiver

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### Abstract

We give some evidences of the AGT-W relation between  $SU(3)$  quiver gauge theories and  $A_2$  Toda theory. In particular, we derive the explicit form of 5-point correlation functions in the lower orders and confirm the agreement with Nekrasov's partition function for  $SU(3) \times SU(3)$  quiver gauge theory. The algorithm to derive the correlation functions can be applied to general  $n$ -point function in  $A_2$  Toda theory which will be useful to establish the relation for more generic quivers. Partial analysis is also given for  $SU(3) \times SU(2)$  case and we comment on some technical issues which need clarification before establishing the relation.

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# 1 Introduction

It is well-known, after the seminal works by Seiberg-Witten [1,2], that there is a close relation between 4-dim  $\mathcal{N} = 2$  gauge theories and the quantum geometry of 2-dim Riemann surface. Recently, Gaiotto [3] invented a new representation of the Seiberg-Witten curve where the duality transformation of the couplings of  $\mathcal{N} = 2$  system is encoded as the duality transformation of the moduli of the curve. More precisely, the hypermultiplet in the  $\mathcal{N} = 2$  gauge theory is represented as a puncture on the curve and the gauge field is given as the cylinder which connects the punctures. The gauge coupling is then extracted as the modulus of the cylinder.

The relation between Gaiotto's curve and  $\mathcal{N} = 2$  gauge theories was further deepened by Alday, Gaiotto and Tachikawa [4]. They observed that the Nekrasov's formula [5] for the partition function of  $\mathcal{N} = 2$  gauge theory coincides with the correlation function of Liouville field theory. They obtained such result for  $N_f = 4$   $SU(2)$  gauge theory (4-point function on sphere in Liouville side) and  $\mathcal{N} = 2^*$   $SU(2)$  gauge theory (1-point function on torus), and conjectured that such relation, which is called “AGT relation”, exists for other superconformal field theories. After this work, many studies have been carried out [6–19], which analytically prove this relation in some cases or limits. This relation also have been studied energetically in the context of Dijkgraaf-Vafa matrix model [20–30]. Moreover, the loop and surface operators in 4-dim gauge theory and their correspondence in Liouville theory are widely discussed [31–41]. The 5-dim extension of this relation also has been studied [42].

A natural generalization of AGT relation is the similar correspondence between  $\mathcal{N} = 2$   $SU(N)$  quiver gauge theories and  $A_{N-1}$  Toda theory [43] (“AGT-W relation”). For  $SU(N)$  case, the puncture in Gaiotto curve has an extra label of Young diagram [3]. For the linear quiver, two punctures with the general Young diagram appear on two edges, while the other punctures are labeled by a special Young diagram  $[N - 1, 1]$ . The latter ones are called “simple punctures.” In [43], by using the example of  $SU(N)$  gauge theory with  $N_f = 2N$ , it was conjectured that the simple puncture is associated with the level-1 singular vector of  $W_N$  algebra, the symmetry of  $A_{N-1}$  Toda field theory. This AGT-W relation is explicitly checked in  $SU(3)$  case up to instanton number 2 [44, 45]. Some analysis for proof in  $SU(N)$  case

also has been done [46–49]. In [50], we generalized this conjecture to the puncture with general Young diagram and determined the possible form of the associated vertex operator. In particular, we confirmed that Gaiotto’s curve can be reproduced through the null state conditions which the vertex operators satisfy in the semi-classical limit.

To establish AGT-W relation with such general punctures, we need develop a method to compute corresponding correlation functions of Toda theory, at least the 5-point functions, which are written in the form of Selberg integral. To obtain an analytic formula to carry out such integration would be highly desirable but at this moment it is technically difficult. As the intermediate step, we will establish, in this paper, an algorithm where partial result (first few terms in the expansion of moduli parameters) can be obtained by computer. For this purpose, we decompose the Riemann surface into the “propagators” and “vertices” as in the perturbative string theory. In particular, for the family of theories which are called linear quiver gauge theory, one needs consider only Riemann sphere as the tree diagrams.

One technical non-triviality in obtaining 3-point vertex for Toda theory (or  $W$ -algebra) is the recursion formula for  $W$ -generators. As discussed in the literature, the conformal Ward identity reduces the number of generators in the correlator through the highest weight conditions. For Toda theory, there remain product of  $W_{-1}$  generators which cannot be simplified further by the recursion formula. It implies that we need some constraints on the primary fields in the correlator to solve it. As shown in [51], a solution is to impose one of the primary fields in the 3-point function to have the level-1 null state condition by which one can replace  $W_{-1}$  by  $L_{-1}$ . For the linear quiver gauge theory, fortunately, if we decompose Gaiotto curve into 3-point functions, we have at least one simple vertex which is indeed characterized by such condition.

We organized the paper as follows. In section 2.1, we summarize Nekrasov’s partition function for linear quiver theory which should be reproduced as the correlation function of Toda theory. In section 2.2, we review the known formulae on the correlation functions of Toda field theory after [51]. In particular, the information of the coefficient of 3-point function is essential to derive the 1-loop part of partition functions. We also review the general strategy of the computation of the correlation functions by decomposing the curve into the “propagators” and “vertices.”

In section 3, we explain the explicit algorithm of the calculation of the conformal blocks in the lower orders. A nontrivial part of this section is the derivation of recursion formula for the 3-point function. As we noted, imposing the level-1 null state condition for one of the operators in the 3-point function is essential to fix their explicit form.

In section 4, we apply our general strategy to a specific quiver gauge theory,  $SU(3) \times SU(3)$  quiver, which corresponds to a 5-point function in Toda theory. We have confirmed the correspondence in both 1-loop and the instanton contribution at the lower levels. We also work on  $SU(3) \times SU(2)$  case. This time, however, we meet some problems. A naive calculation of the 5 point function shows that it vanishes automatically. In order to proceed further, we removing the factor which vanishes by hand. After that, the remaining factors reproduce some part of the Nekrasov's partition function. However, the correspondence is not complete yet. We argue that we need to tune some parameters (such as the mass of the bifundamental matter field) in order to meet the gauge side. We doubt that the correlation function of Toda theory may need some modification when two of the vertex operators in 3-point function have null states at level one. We give some preliminary analysis in appendix D. We hope to clarify this in the future work.

## 2 AGT-W relation

AGT relation [4] reveals the nontrivial correspondence between the partition function of 4-dim  $\mathcal{N} = 2$   $SU(2)$  quiver gauge theory and the correlation function of 2-dim Liouville (or  $A_1$  Toda) field theory. The 2-dim theory is defined on Seiberg-Witten curve which determines the field contents of the 4-dim theory. The Seiberg-Witten system can be interpreted as the intersecting D4/NS5-branes' system, where the intersection points exist on Seiberg-Witten curve [3, 52]. AGT relation says that when we insert the Liouville vertex operators at all the intersection points and calculate the correlation function of these vertex operators in the 2-dim theory, this function can be rewritten as the product of the partition function of the 4-dim theory and some additional factors. Schematically, it is written as

$$|Z^{\text{gauge}}|^2 = \langle V \cdots V \rangle^{\text{Liouville}} \quad (1)$$

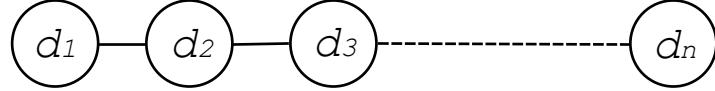


Figure 1: Linear quiver gauge theory

where  $Z^{\text{gauge}}$  is the partition function of quiver gauge theory. In the right hand side, we have the conformal block of Liouville theory where  $V$  is the vertex operator insertion.

A natural generalization of AGT relation seems a similar correspondence between the partition function of 4-dim  $\mathcal{N} = 2$   $SU(N)$  quiver gauge theory and the correlation function of 2-dim  $A_{N-1}$  Toda field theory [43]. In this paper, we examine this conjecture for  $N = 3$  case, that is, the case of  $SU(3)$  quiver gauge theory and  $A_2$  Toda field theory. First, in this section, we review the form of the functions on the both sides.

## 2.1 4-dim gauge theory side: partition functions

The partition function of 4-dim  $\mathcal{N} = 2$  gauge theory is written as the product of classical part, 1-loop correction part and nonperturbative instanton correction part:

$$Z^{\text{gauge}} = Z_{\text{class}} Z_{\text{1-loop}} Z_{\text{inst}} . \quad (2)$$

In the following, we consider the case of  $\prod_{k=1}^n SU(d_k)$  linear quiver gauge theory (fig. 1). Especially, we will consider only the conformal invariant cases where  $k_a = 2d_a - d_{a+1} - d_{a-1} (\geq 0)$  fundamental hypermultiplets are attached to gauge group  $SU(d_a)$ .

The classical part of the partition function is

$$Z_{\text{class}} = \exp \left[ \sum_{k=1}^n 2\pi i \tau_k |\vec{a}_k|^2 \right] , \quad (3)$$

where  $\tau_k := \frac{\theta_k}{2\pi} + \frac{4\pi i}{g_k^2}$  is the complex UV coupling constant.  $\vec{a}_k := \sum_{i=1}^{d_k-1} a_i \vec{e}_i$  is the diagonal of VEV's  $a_i$  of adjoint scalars, where  $\vec{e}_i$  are the simple roots of gauge symmetry algebra. For  $SU(d)$  algebra, we usually set

$$\vec{e}_i = (0, \dots, 0, 1, -1, 0, \dots, 0) , \quad (4)$$

where 1 is  $i$ -th element. It gives, for example,  $\vec{a} = (a_1, -a_1)$  for  $SU(2)$  and  $\vec{a} = (a_1, -a_1 + a_2, -a_2)$  for  $SU(3)$ .

The 1-loop contribution to the partition function is [4, 43]

$$\begin{aligned} Z_{\text{1-loop}} &= \left( \prod_{k=1}^n z_{\text{vec}}^{\text{1lp}}(\vec{a}_k) \right) \left( \prod_{\bar{p}=1}^{d_1} z_{\text{afd}}^{\text{1lp}}(\vec{a}_1, \bar{\mu}_{\bar{p}}) \right) \\ &\quad \times \left( \prod_{k=1}^{n-1} z_{\text{bfd}}^{\text{1lp}}(\vec{a}_k, \vec{a}_{k+1}, m_k) \right) \left( \prod_{p=1}^{d_n} z_{\text{fd}}^{\text{1lp}}(\vec{a}_n, \mu_p) \right), \end{aligned} \quad (5)$$

where  $\mu_p, \bar{\mu}_{\bar{p}}, m_k$  are the mass of fundamental, antifundamental, bifundamental fields, respectively, and

$$\begin{aligned} z_{\text{vec}}^{\text{1lp}}(\vec{a}) &= \prod_{i < j} \exp [-\gamma_{\epsilon_1, \epsilon_2}(\hat{a}_i - \hat{a}_j - \epsilon_1) - \gamma_{\epsilon_1, \epsilon_2}(\hat{a}_i - \hat{a}_j - \epsilon_2)], \\ z_{\text{fd}}^{\text{1lp}}(\vec{a}, \mu) &= \prod_i \exp [\gamma_{\epsilon_1, \epsilon_2}(\hat{a}_i - \mu)], \\ z_{\text{afd}}^{\text{1lp}}(\vec{a}, \bar{\mu}) &= \prod_i \exp [\gamma_{\epsilon_1, \epsilon_2}(-\hat{a}_i + \bar{\mu} - \epsilon_+)], \\ z_{\text{bfd}}^{\text{1lp}}(\vec{a}, \vec{b}, m) &= \prod_{i,j} \exp \left[ \gamma_{\epsilon_1, \epsilon_2}(\hat{a}_i - \hat{b}_j - m) \right], \end{aligned} \quad (6)$$

where  $\epsilon_+ := \epsilon_1 + \epsilon_2$  ( $\epsilon_1, \epsilon_2$  are Nekrasov's deformation parameters), and the function  $\gamma_{\epsilon_1, \epsilon_2}(x)$  is related to double Gamma function  $\Gamma_2(x|\epsilon_1, \epsilon_2)$  as<sup>1</sup>

$$\gamma_{\epsilon_1, \epsilon_2}(x) = \log \Gamma_2(x + \epsilon_+ | \epsilon_1, \epsilon_2). \quad (7)$$

The properties of double Gamma function are summarized in appendix A.

The instanton contribution is obtained by Nekrasov's instanton counting formula with Young tableaux [4, 13]:

$$\begin{aligned} Z_{\text{inst}} &= \sum_{\{\vec{Y}_1, \dots, \vec{Y}_n\}} \left( \prod_{k=1}^n q_k^{|\vec{Y}_k|} z_{\text{vec}}(\vec{a}_k, \vec{Y}_k) \right) \left( \prod_{\bar{p}=1}^{d_1} z_{\text{afd}}(\vec{a}_1, \vec{Y}_1, \bar{\mu}_{\bar{p}}) \right) \\ &\quad \times \left( \prod_{k=1}^{n-1} z_{\text{bfd}}(\vec{a}_k, \vec{Y}_k; \vec{a}_{k+1}, \vec{Y}_{k+1}; m_k) \right) \left( \prod_{p=1}^{d_n} z_{\text{fd}}(\vec{a}_n, \vec{Y}_n, \mu_p) \right), \end{aligned} \quad (8)$$

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<sup>1</sup> We would like to thank Satoshi Nawata for pointing out typos in the previous version: the definition of  $\gamma_{\epsilon_1, \epsilon_2}(x)$  in eq. (7) and the term for  $\vec{\lambda}_i$  just above eq. (22).

where  $q_k := e^{2\pi i \tau_k}$  ( $\tau_k$  is the coupling constant), and  $\vec{Y}_k = (Y_{k,1}, \dots, Y_{k,d_k})$  is a set of Young tableaux.  $|\vec{Y}_k|$  is the total sum of number of boxes of Young tableaux  $Y_{k,i}$  ( $i = 1, \dots, d_k$ ). Each factor of the instanton part is written as

$$\begin{aligned} z_{\text{bfd}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; m) &= \prod_{i,j} \prod_{s \in Y_i} (E(\hat{a}_i - \hat{b}_j, Y_i, W_j, s) - m) \\ &\quad \times \prod_{t \in W_j} (\epsilon_+ - E(\hat{b}_j - \hat{a}_i, W_j, Y_i, t) - m), \\ z_{\text{vec}}(\vec{a}, \vec{Y}) &= 1/z_{\text{bfd}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}; 0), \\ z_{\text{fd}}(\vec{a}, \vec{Y}, \mu) &= \prod_i \prod_{s \in Y_i} (\phi(\hat{a}_i, s) - \mu + \epsilon_+), \\ z_{\text{afd}}(\vec{a}, \vec{Y}, \bar{\mu}) &= z_{\text{fd}}(\vec{a}, \vec{Y}, \epsilon_+ - \bar{\mu}). \end{aligned} \tag{9}$$

The functions  $E(\hat{a}, Y, W, s)$  and  $\phi(\hat{a}, s)$  are defined as

$$\begin{aligned} E(\hat{a}, Y, W, s) &= \hat{a} - \epsilon_1(\lambda'_{W,j} - i) + \epsilon_2(\lambda_{Y,i} - j + 1), \\ \phi(\hat{a}, s) &= \hat{a} + \epsilon_1(i - 1) + \epsilon_2(j - 1), \end{aligned} \tag{10}$$

where  $s = (i, j)$  denotes the position of the box in a Young tableau (*i.e.* the box in  $i$ -th column and  $j$ -th row).  $\lambda_{Y,i}$  is the height of  $i$ -th column, and  $\lambda'_{Y,j}$  is the length of  $j$ -th row for Young tableau  $Y$ . That is,  $\lambda'_{Y,j} - i$  and  $\lambda_{Y,i} - j$  are the length of ‘leg’ and ‘arm’ of the Young tableau  $Y$  for the box  $s = (i, j)$ , respectively.

## 2.2 2-dim CFT: W-algebra, Toda theory, and correlation functions

The action of 2-dim  $A_{N-1}$  Toda field theory is

$$S = \int d^2\sigma \sqrt{g} \left[ \frac{1}{8\pi} g^{xy} \partial_x \vec{\varphi} \cdot \partial_y \vec{\varphi} + \mu \sum_{k=1}^{N-1} e^{b\vec{e}_k \cdot \vec{\varphi}} + \frac{Q}{4\pi} R \vec{\rho} \cdot \vec{\varphi} \right], \tag{11}$$

where  $\vec{\varphi} = (\varphi_1, \dots, \varphi_N)$  is the Toda fields satisfying  $\sum \varphi_k = 0$ .  $g_{xy}$  is the metric on 2-dim Riemann surface, and  $R$  is its curvature.  $\vec{e}_k$  is the  $k$ -th simple root written as eq. (4), and  $\vec{\rho}$  is the Weyl vector (*i.e.* half the sum of all positive roots) of  $A_{N-1}$  algebra.  $b$  is a real parameter, and  $Q := b + 1/b$ . This theory is conformal invariant with the central charge

$$c = (N-1) + 12Q^2 \vec{\rho} \cdot \vec{\rho} = (N-1)(1 + N(N+1)Q^2). \tag{12}$$

The symmetry algebra of this theory is generated by the energy-momentum tensor  $T(z)$  and additional  $N - 2$  chiral currents  $W^{(3)}, \dots, W^{(N)}$  with spin  $3, \dots, N$ . In the following in this paper, we concentrate on  $N = 3$  case. In this case, the following generators are defined as Laurent expansion of the currents:

$$T(z) =: \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}, \quad W^{(3)}(z) =: \sum_{n=-\infty}^{\infty} \frac{W_n}{z^{n+3}}. \quad (13)$$

The commutation relation for the generators is given by

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ [L_n, W_m] &= (2n - m)W_{n+m} \\ \frac{2}{9}[W_n, W_m] &= \frac{c}{3 \cdot 5!}n(n^2 - 1)(n^2 - 4)\delta_{n+m,0} + \frac{16}{22 + 5c}(n - m)\Lambda_{n+m} \\ &\quad + (n - m)\left(\frac{1}{15}(n + m + 2)(n + m + 3) - \frac{1}{6}(n + 2)(m + 2)\right)L_{n+m}, \end{aligned} \quad (14)$$

where

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5}x_n L_n, \quad (15)$$

with  $x_{2l} = (1 + l)(1 - l)$  and  $x_{2l+1} = (2 + l)(1 - l)$ .

In the following, we need the action of such generators on the operators at arbitrary point  $z = \zeta$ . For this purpose, it is useful to introduce the operators  $L_n(\zeta)$  and  $W_n(\zeta)$  defined by the contour integration around  $z = \zeta$ ,

$$L_n(\zeta) = \oint_{z=\zeta} \frac{dz}{2\pi i} (z - \zeta)^{n+1} T(z), \quad W_n(\zeta) = \oint_{z=\zeta} \frac{dz}{2\pi i} (z - \zeta)^{n+2} W(z). \quad (16)$$

The operators in eq.(13) are identical to the special case  $\zeta = 0$ , namely  $L_n = L_n(0)$  and  $W_n = W_n(0)$ . The commutation relations among  $L_n(\zeta)$  and  $W_n(\zeta)$  are identical to eq.(14).

The highest weight (ket) state  $|\Delta, w\rangle$  and its conjugate (bra) state  $\langle\Delta, w|$  is defined by the conditions:

$$\begin{aligned} L_n|\Delta, w\rangle &= 0, & W_n|\Delta, w\rangle &= 0, & (n > 0) \\ L_0|\Delta, w\rangle &= \Delta|\Delta, w\rangle, & W_0|\Delta, w\rangle &= w|\Delta, w\rangle, \end{aligned} \quad (17)$$

$$\begin{aligned} \langle\Delta, w|L_n &= 0, & \langle\Delta, w|W_n &= 0, & (n < 0) \\ \langle\Delta, w|L_0 &= \langle\Delta, w|\Delta, & \langle\Delta, w|W_0 &= \langle\Delta, w|w. \end{aligned} \quad (18)$$

The adjoint of operators are defined as

$$L_n^\dagger = L_{-n}, \quad W_n^\dagger = W_{-n}. \quad (19)$$

In terms of Toda fields, the highest weight state is given by the vertex operator:

$$\begin{aligned} V_{\vec{\alpha}}(z) &=: e^{\vec{\alpha} \cdot \vec{\varphi}(z)} : \quad (\vec{\alpha} \in \mathbf{C}^3, \quad \sum_{i=1}^3 \alpha_i = 0), \\ |V_{\vec{\alpha}}\rangle &= \lim_{z \rightarrow 0} V_{\vec{\alpha}}(z)|0\rangle, \quad \langle V_{\vec{\alpha}}| = \lim_{z \rightarrow \infty} z^{2\Delta_{\vec{\alpha}}} \langle 0|V_{\vec{\alpha}}(z). \end{aligned} \quad (20)$$

The parameters  $\Delta, w$  are related to  $\vec{\alpha}$  as

$$\Delta_{\vec{\alpha}} = \frac{1}{2}(2Q\vec{\rho} - \vec{\alpha}) \cdot \vec{\alpha}, \quad w_{\vec{\alpha}} = i\sqrt{\frac{48}{22+5c}} \prod_{i=1}^3 (\vec{\alpha} - Q\vec{\rho}) \cdot \vec{\lambda}_i, \quad (21)$$

where  $\vec{\lambda}_i$  are the weights of the fundamental representation of  $A_2$  Lie algebra:

$$\begin{aligned} \vec{\lambda}_1 &= \frac{1}{3}(2, -1, -1), \quad \vec{\lambda}_2 = \vec{\lambda}_1 - \vec{e}_1 = \frac{1}{3}(-1, 2, -1), \\ \vec{\lambda}_3 &= \vec{\lambda}_2 - \vec{e}_2 = \frac{1}{3}(-1, -1, 2). \end{aligned} \quad (22)$$

and  $\vec{\rho} = (1, 0, -1)$  is the Weyl vector.

The inner product  $\langle V_{\vec{\alpha}} | V_{\vec{\alpha}} \rangle$  vanishes in general because of the conservation of momentum. The nontrivial inner product can be taken in the form  $\langle V_{2Q\vec{\rho}-\vec{\alpha}} | V_{\vec{\alpha}} \rangle = 1$ .

### Correlation functions

It is known that the form of general 3-point functions of primary fields is determined by the conformal invariance as [51]

$$\langle V_{\vec{\alpha}_1}(z_1) V_{\vec{\alpha}_2}(z_2) V_{\vec{\alpha}_3}(z_3) \rangle = \frac{C(\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3)}{|z_{12}|^{2(\Delta_1 + \Delta_2 - \Delta_3)} |z_{13}|^{2(\Delta_1 + \Delta_3 - \Delta_2)} |z_{23}|^{2(\Delta_2 + \Delta_3 - \Delta_1)}}, \quad (23)$$

where  $z_{ij} := z_i - z_j$  and  $\Delta_i := \Delta_{\vec{\alpha}_i}$ . It is difficult to calculate the coefficients  $C(\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3) = \langle V_{\vec{\alpha}_1} | V_{\vec{\alpha}_2}(1) | V_{\vec{\alpha}_3} \rangle$  for general  $\vec{\alpha}_i$ , however, if one of  $\vec{\alpha}_i$ 's is

proportional to  $\vec{\lambda}_1$  or  $-\vec{\lambda}_3$  (the highest weight of fundamental or antifundamental representation), they are obtained as

$$C(\vec{\alpha}_1, \vec{\alpha}_2, \gamma \vec{\lambda}_1) = \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{(2Q\vec{\rho}-\sum \vec{\alpha}_i) \cdot \vec{\rho}/b} \times \frac{\Upsilon(b)^2 \Upsilon(\gamma) \prod_{e>0} \Upsilon((Q\vec{\rho} - \vec{\alpha}_1) \cdot \vec{e}) \Upsilon((Q\vec{\rho} - \vec{\alpha}_2) \cdot \vec{e})}{\prod_{i,j} \Upsilon(\frac{1}{3}\gamma - (\vec{\alpha}_1 - Q\vec{\rho}) \cdot \vec{\lambda}_i - (\vec{\alpha}_2 - Q\vec{\rho}) \cdot \vec{\lambda}_j)}, \quad (24)$$

and

$$C(\vec{\alpha}_1, \vec{\alpha}_2, -\gamma \vec{\lambda}_3) = \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{(2Q\vec{\rho}-\sum \vec{\alpha}_i) \cdot \vec{\rho}/b} \times \frac{\Upsilon(b)^2 \Upsilon(\gamma) \prod_{e>0} \Upsilon((Q\vec{\rho} - \vec{\alpha}_1) \cdot \vec{e}) \Upsilon((Q\vec{\rho} - \vec{\alpha}_2) \cdot \vec{e})}{\prod_{i,j} \Upsilon(\frac{1}{3}\gamma + (\vec{\alpha}_1 - Q\vec{\rho}) \cdot \vec{\lambda}_i + (\vec{\alpha}_2 - Q\vec{\rho}) \cdot \vec{\lambda}_j)}, \quad (25)$$

where  $\gamma$  is a constant, and  $\prod_{e>0}$  means the product over all positive roots.  $\Upsilon$  is the Upsilon function whose properties are summarized in appendix A.

In order to calculate general  $n$ -point functions, we often use the following decomposition [53]:

$$\begin{aligned} & \langle \mathcal{O}_n(z_n), \dots, \mathcal{O}_3(z_3), \mathcal{O}_2(z_2), \mathcal{O}_1(z_1) \rangle \\ &= \sum_{\vec{\alpha}, Y_1, Y_2} (z_2 - z_1)^{\Delta_V - \Delta_{\mathcal{O}_2} - \Delta_{\mathcal{O}_1}} \langle \mathcal{O}_n(z_n), \dots, \mathcal{O}_3(z_3), \mathcal{L}_{-Y_1}(z_1) V_{\vec{\alpha}}(z_1) \rangle \\ & \quad \times (S_{\vec{\alpha}}^{-1})_{Y_1, Y_2} \langle V_{2Q\vec{\rho} - \vec{\alpha}} | (\mathcal{L}_{-Y_2})^\dagger \mathcal{O}_2(1) | \mathcal{O}_1 \rangle, \end{aligned} \quad (26)$$

where the index  $\vec{\alpha}$  labels the primary fields and  $Y_1, Y_2$  labels the descendants of the primary fields, such as

$$\mathcal{L}_{-Y}(z) V_{\vec{\alpha}}(z) := L_{-n_1}(z) \cdots L_{-n_l}(z) W_{-n'_1}(z) \cdots W_{-n'_w}(z) V_{\vec{\alpha}}(z). \quad (27)$$

Here  $Y$  is a set of two Young tableaux  $(Y_L, Y_W)$  with  $Y_L = (n_1 \geq \dots \geq n_l)$  and  $Y_W = (n'_1 \geq \dots \geq n'_w)$ . It means that  $|Y| := \sum_{i=1}^l n_i + \sum_{j=1}^w n'_j$  is the level of the descendant. The descendants at the same level  $|Y|$  can be labeled by the partitions of integer  $|Y|$ , so it is useful to classify them using the sets of Young tableaux  $Y$ . The conformal dimension  $\Delta_V$  in eq. (26) is that of  $\mathcal{L}_{-Y_2} V_{\vec{\alpha}}$  which is defined in this way.

The matrix  $S$  is called as Shapovalov matrix

$$S_{\vec{\alpha}, Y_1, Y_2} := \langle V_{2Q\vec{\rho} - \vec{\alpha}} | (\mathcal{L}_{-Y_1})^\dagger \mathcal{L}_{-Y_2} | V_{\vec{\alpha}} \rangle \quad (28)$$

Once  $Y_1, Y_2$  is given, it is possible to determine Shapovalov matrix by using the commutation relations (13) and the highest weight conditions (17), (18). We note that unless  $|Y_1| = |Y_2|$ , the inner product vanishes. It implies that the Shapovalov matrix is block diagonal and each block is finite size. It helps us to determine the inverse Shapovalov matrix by restricting the computation to each level. The explicit form of the Shapovalov matrix for lower levels is given in appendix B and [44].

When we decompose the  $n$ -point functions for vertex operators using such rule repeatedly, they necessarily become the linear combination of the products of inverse Shapovalov matrices (or ‘‘propagators’’)  $S^{-1}$  and 3-point functions (or ‘‘vertices’’). In the decomposition (26), together with Shapovalov matrix, we have correlation function of the form

$$\begin{aligned} \Gamma_{\vec{\alpha}_\infty, \vec{\alpha}_1, \vec{\alpha}_0}(Y_\infty, Y_1, Y_0) \\ := \langle V_{2Q\vec{\rho}-\vec{\alpha}_\infty} | (\mathcal{L}_{-Y_\infty})^\dagger \mathcal{L}_{-Y_1}(1) V_{\vec{\alpha}_1}(1) \mathcal{L}_{-Y_0} | V_{\vec{\alpha}_0} \rangle \\ =: \langle \mathcal{L}_{-Y_\infty} V_{2Q\vec{\rho}-\vec{\alpha}_\infty}, \mathcal{L}_{-Y_1} V_{\vec{\alpha}_1}, \mathcal{L}_{-Y_0} V_{\vec{\alpha}_0} \rangle. \end{aligned} \quad (29)$$

We note that in this expression, the operator at infinity is defined by the adjoint (19). By repeating the decomposition, we are left with a 3-point function in the leftmost factor. Here we have to be careful in the definition of operator at  $z = \infty$ . In order to keep conformal property, we have to use the conformal transformation, say  $w = 1/z$ , to define the adjoint (BPZ conjugation). For Virasoro generators, the two definitions agree. For  $W$ -operator, however, BPZ conjugation ( $W^\flat$ ) is given as, by using  $W(z) = W(w)(\frac{dw}{dz})^3 = -W(w)w^6$ ,

$$(W_n)^\flat = -W_{-n} = -W_n^\dagger. \quad (30)$$

Thus we have discrepancy in the sign. In order to describe the 3-point function which appears in the leftmost factor, we need to introduce another type of 3-point function  $\Gamma'$ :

$$\begin{aligned} \Gamma'_{\vec{\alpha}_\infty, \vec{\alpha}_1, \vec{\alpha}_0}(Y_\infty, Y_1, Y_0) \\ := \langle V_{\vec{\alpha}_\infty} | (\mathcal{L}_{-Y_\infty})^\flat \mathcal{L}_{-Y_1}(1) V_{\vec{\alpha}_1}(1) \mathcal{L}_{-Y_0} | V_{\vec{\alpha}_0} \rangle \\ =: \langle \mathcal{L}_{-Y_\infty} V_{\vec{\alpha}_\infty}, \mathcal{L}_{-Y_1} V_{\vec{\alpha}_1}, \mathcal{L}_{-Y_0} V_{\vec{\alpha}_0} \rangle'. \end{aligned} \quad (31)$$

where  $\mathcal{L}_{-Y_\infty}(\infty)$  in the last line to be defined by BPZ conjugation. Again, note that  $\Gamma'$  is different from  $\Gamma$  in the sign.

To summarize, the  $n$ -point functions can be written in the form:

$$\langle \mathcal{O}_n, \dots, \mathcal{O}_3, \mathcal{O}_2, \mathcal{O}_1 \rangle \sim \sum \Gamma' S^{-1} \Gamma S^{-1} \Gamma \dots \Gamma S^{-1} \Gamma. \quad (32)$$

When all the vertices in the correlator  $\mathcal{O}_i$  are primary, one needs only the special case where  $Y_1 = \emptyset$  (no boxes) in all the 3-point functions  $\Gamma, \Gamma'$ .

If one can determine these 3-point functions, one can in principle calculate the general  $n$ -point correlation functions by patching them together with the inverse Shapovalov matrices. As we will see in the next section, this is indeed possible as long as one of the vertex operators in 3-point functions satisfies  $\vec{\alpha}_i \propto \vec{\lambda}_1$  or  $\vec{\alpha}_i \propto -\vec{\lambda}_3$ . We note that such operators satisfy the null state condition at level-1, which is necessary to reduce the degree of freedom of  $W$ -generators. The above is our strategy to obtain the  $n$ -point functions.

### 3 Recursion formula for 3-point functions

In order to determine the 3-point functions (29) and (31), we use the deformation of integration contour paths and derive the recursion formula. In the remainder of this section, we write  $\tilde{V}_\infty := \mathcal{L}_{-Y_\infty} V_{\vec{\alpha}_\infty}(\infty)$ ,  $\tilde{V}_1 := \mathcal{L}_{-Y_1} V_{\vec{\alpha}_1}(1)$ ,  $\tilde{V}_0 := \mathcal{L}_{-Y_0} V_{\vec{\alpha}_0}(0)$ , for simplicity.

#### Recursion formula for $T$ insertion

First, the standard CFT recursion formula for  $\Gamma$ -type 3-point functions (29) gives

$$\begin{aligned} \langle L_{-n} \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle &= \oint_\infty \frac{dz}{2\pi i} z^{n+1} \langle T(z) \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle \\ &= \oint_1 \frac{dz}{2\pi i} \sum_k \frac{z^{n+1}}{(z-1)^{k+2}} \langle \tilde{V}_\infty, L_k \tilde{V}_1, V_0 \rangle + \oint_0 \frac{dz}{2\pi i} \sum_k \frac{z^{n+1}}{z^{k+2}} \langle \tilde{V}_\infty, \tilde{V}_1, L_k \tilde{V}_0 \rangle \\ &= \langle \tilde{V}_\infty, [L_{-1} + (n+1)L_0 + \sum_{i=1}^n \frac{(n+1)!}{(i+1)!(n-i)!} L_i] \tilde{V}_1, \tilde{V}_0 \rangle + \langle \tilde{V}_\infty, \tilde{V}_1, L_n \tilde{V}_0 \rangle. \end{aligned} \quad (33)$$

In particular, for  $n = 0$  case, we obtain

$$\langle \tilde{V}_\infty, L_{-1} \tilde{V}_1, \tilde{V}_0 \rangle = \langle L_0 \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle - \langle \tilde{V}_\infty, L_0 \tilde{V}_1, \tilde{V}_0 \rangle - \langle \tilde{V}_\infty, \tilde{V}_1, L_0 \tilde{V}_0 \rangle. \quad (34)$$

Using this relation, we can remove the  $L_{-1}$  operator in eq. (33):

$$\begin{aligned} \langle L_{-n}\tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle &= \langle L_0\tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle + \langle \tilde{V}_\infty, \tilde{V}_1, (L_n - L_0)\tilde{V}_0 \rangle \\ &\quad + \langle \tilde{V}_\infty, (nL_0 + \sum_{i=1}^n \frac{(n+1)!}{(i+1)!(n-i)!} L_i)\tilde{V}_1, \tilde{V}_0 \rangle. \end{aligned} \quad (35)$$

On the right hand side, there are only the operators  $L_n$  with non-negative  $n$ . So, use of the highest weight condition and the commutation relations will simplify the expression further.

In similar ways, we can derive the recursion formulae for the action of  $L_{-n}$  on  $\tilde{V}_1$  and  $\tilde{V}_0$ :

$$\begin{aligned} \langle \tilde{V}_\infty, \tilde{V}_1, L_{-n}\tilde{V}_0 \rangle &= \langle (L_n - L_0)\tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle + \langle \tilde{V}_\infty, \tilde{V}_1, L_0\tilde{V}_0 \rangle \\ &\quad + \langle \tilde{V}_\infty, (nL_0 + \sum_{i=1}^\infty \frac{(-1)^i(n-1+i)!}{(i+1)!(n-2)!} L_i)\tilde{V}_1, \tilde{V}_0 \rangle, \end{aligned} \quad (36)$$

$$\begin{aligned} \langle \tilde{V}_\infty, L_{-n}\tilde{V}_1, \tilde{V}_0 \rangle &= \langle [(-1)^{n+1}(L_0 - L_1) + L_n + \sum_{j=0}^\infty \frac{(n-1+j)!}{(j+1)!(n-2)!} L_{n+1+j}] \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle \\ &\quad + (-1)^n \langle \tilde{V}_\infty, \tilde{V}_1, (nL_0 + \sum_{i=1}^\infty \frac{(n-1+i)!}{(i+1)!(n-2)!} L_i)\tilde{V}_0 \rangle + (-1)^n \langle \tilde{V}_\infty, L_0\tilde{V}_1, \tilde{V}_0 \rangle. \end{aligned} \quad (37)$$

### Recursion formula for $W$ insertion

We start again from the standard conformal bootstrap for  $\Gamma$ -type 3-point function:

$$\begin{aligned} \langle W_{-n}\tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle &= \langle \tilde{V}_\infty, [W_{-2} + (n+2)W_{-1} + \frac{1}{2}(n+1)(n+2)W_0 + \sum_{i=1}^n \frac{(n+2)!}{(i+2)!(n-i)!} W_i] \tilde{V}_1, \tilde{V}_0 \rangle \\ &\quad + \langle \tilde{V}_\infty, \tilde{V}_1, W_n\tilde{V}_0 \rangle. \end{aligned} \quad (38)$$

In particular, for  $n = 0$  case, we obtain

$$\begin{aligned} \langle \tilde{V}_\infty, W_{-2}\tilde{V}_1, \tilde{V}_0 \rangle &= \langle W_0\tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle - \langle \tilde{V}_\infty, W_0\tilde{V}_1, \tilde{V}_0 \rangle - \langle \tilde{V}_\infty, \tilde{V}_1, W_0\tilde{V}_0 \rangle \\ &\quad - 2\langle \tilde{V}_\infty, W_{-1}\tilde{V}_1, \tilde{V}_0 \rangle. \end{aligned} \quad (39)$$

As in the recursion formula for  $T$  insertion, we combine it with the original general formula (38), then

$$(38) \quad = \quad \langle W_0 \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle + \langle \tilde{V}_\infty, \tilde{V}_1, (W_n - W_0) \tilde{V}_0 \rangle \\ + \langle \tilde{V}_\infty, [nW_{-1} + \frac{1}{2}n(n+3)W_0 + \sum_{i=1}^n \frac{(n+2)!}{(i+2)!(n-i)!} W_i] \tilde{V}_1, \tilde{V}_0 \rangle. \quad (40)$$

This is simpler than the original one, since we don't have  $W_{-2}$  insertion. On the other hand, unlike the  $T$  insertion, there remains  $W_{-1}$  insertion which makes the analysis fundamentally difficult. In fact, as we will see later, we need impose the level-1 null state condition on one of the vertex operators to solve the recursion.

Similarly, the formula for insertion of  $W_{-n}$  in  $\tilde{V}_1$  or  $\tilde{V}_0$  is given as

$$\langle \tilde{V}_\infty, \tilde{V}_1, W_{-n} \tilde{V}_0 \rangle = \langle (W_n - W_0) \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle + \langle \tilde{V}_\infty, \tilde{V}_1, W_0 \tilde{V}_0 \rangle \\ + \langle \tilde{V}_\infty, [nW_{-1} - \frac{1}{2}n(n-3)W_0 - \sum_{i=1}^{\infty} \frac{(-1)^i(n-1+i)!}{(i+2)!(n-3)!} W_i] \tilde{V}_1, \tilde{V}_0 \rangle, \quad (41)$$

$$\langle \tilde{V}_\infty, W_{-n} \tilde{V}_1, \tilde{V}_0 \rangle \\ = \langle [W_n + (n-2)W_{n+1} + \sum_{j=0}^{\infty} \frac{(n-1+j)!}{(j+2)!(n-3)!} W_{n+2+j}] \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle \\ + (-1)^n \langle [(n-1)W_0 - (n-2)W_1 - W_2] \tilde{V}_\infty, \tilde{V}_1, \tilde{V}_0 \rangle \\ - (-1)^n \langle \tilde{V}_\infty, \tilde{V}_1, [\frac{1}{2}n(n-1)W_0 + \sum_{i=1}^{\infty} \frac{(n-1+i)!}{(i+2)!(n-3)!} W_i] \tilde{V}_0 \rangle \\ - (-1)^n \langle \tilde{V}_\infty, [nW_{-1} + (n-1)W_0] \tilde{V}_1, \tilde{V}_0 \rangle. \quad (42)$$

### General procedure

The recursion formula obtained so far will give the following algorithm to compute the 3-point correlation functions which take the general form of

$$\langle \mathcal{L}_{-Y_\infty} V_{\vec{\alpha}_\infty}, \mathcal{L}_{-Y_1} V_{\vec{\alpha}_1}, \mathcal{L}_{-Y_0} V_{\vec{\alpha}_0} \rangle \quad (43)$$

where  $V$ 's are the primary fields. It is given by repeating following steps:

1. Scan the leftmost operators acting on each primary field. If you find an operator of the form  $L_n$  ( $n < 0$ ) or  $W_n$  ( $n < -1$ ), apply one of the

appropriate formula in eqs. (34)–(37) and eqs. (39)–(42) which adds the leftmost operator  $L_n$  ( $n \geq 0$ ) or  $W_n$  ( $n \geq -1$ ) to different entries in the 3-point function.

2. Apply the commutation relation (14) to make each entry normal ordering, and the highest weight condition (17) for each vertex operator.

In each step, the degree of operators  $|Y_\infty| + |Y_1| + |Y_0|$  decreases monotonously. In the end, eq. (43) becomes the linear combination of functions of the form

$$\langle V_{\vec{\alpha}_\infty}, (W_{-1})^\ell V_{\vec{\alpha}_1}, V_{\vec{\alpha}_0} \rangle \quad (\ell = 0, 1, 2, \dots) \quad (44)$$

with the coefficients depending on  $\Delta$  and  $w$  of the vertex operators.

For the generic  $V_{\vec{\alpha}_\infty}$ ,  $V_{\vec{\alpha}_1}$ ,  $V_{\vec{\alpha}_0}$ , this is the dead end of the recursion formula. That is, the expressions (44) with  $\ell = 1, 2, \dots$  remain undetermined, and the correlation function is generally written in terms of these quantities.

However, for the computation of the correlation functions of Toda theory associated with the linear quiver gauge theory, we have the extra condition that one of the vertex operators, say  $V_{\vec{\alpha}_1}$ , satisfies  $\vec{\alpha}_1 = \gamma \vec{\lambda}_1$  or  $-\gamma \vec{\lambda}_3$ . The vertex operator of this form corresponds to the ‘simple’ puncture on Seiberg-Witten curve, and it is the level-1 degenerate state which satisfies

$$W_{-1} V_{\vec{\alpha}_1} = \frac{3w_{\vec{\alpha}_1}}{2\Delta_{\vec{\alpha}_1}} L_{-1} V_{\vec{\alpha}_1}. \quad (45)$$

Use of this condition reduces the number of  $W_{-1}$  in the correlator (44). After we use the above reduction algorithm again and again, all the  $W_{-1}$  operators can be removed in the end. Then we are left with the linear combination of the single correlator  $\langle V_{\vec{\alpha}_\infty}, V_{\vec{\alpha}_1}, V_{\vec{\alpha}_0} \rangle$  whose coefficients are some functions of  $\Delta$  and  $w$ . The concrete form of this correlator is already given in eqs. (23)–(25).

It means that by using this algorithm, we can evaluate general 3-point functions which appear in the following discussion.

## 4 Check on AGT-W relation

In the previous section, we study the general procedure to calculate  $(n+3)$ -point correlation functions in  $A_2$  Toda theory. Their general form is

$$\begin{aligned} & \left\langle V_{\vec{\beta}_\infty}(\infty) V_{\vec{\beta}_{n+1}}(1) \left( \prod_{k=1}^n V_{\vec{\beta}_k}(q_1 \cdots q_k) \right) V_{\vec{\beta}_0}(0) \right\rangle \\ &= \sum_{\{\vec{\alpha}_k\}} \sum_{|Y_k|=|Y'_k|} \left( \prod_{k=1}^n q_k^{\Delta_{\vec{\alpha}_k}} \right) \left( \prod_{k=1}^n (q_1 \cdots q_k)^{-\Delta_{\vec{\beta}_k}} (q_1 \cdots q_n)^{-\Delta_{\vec{\beta}_0}} \right) \\ & \quad \times \Gamma'_{\vec{\beta}_\infty, \vec{\beta}_{n+1}, \vec{\alpha}_1}(\emptyset, \emptyset, Y_1) \left( \prod_{k=1}^{n-1} q_k^{|Y_k|} S_{\vec{\alpha}_k}^{-1}(Y_k, Y'_k) \Gamma_{\vec{\alpha}_k, \vec{\beta}_k, \vec{\alpha}_{k+1}}(Y'_k, \emptyset, Y_{k+1}) \right) \\ & \quad \times q_n^{|Y_n|} S_{\vec{\alpha}_n}^{-1}(Y_n, Y'_n) \Gamma_{\vec{\alpha}_n, \vec{\beta}_n, \vec{\beta}_0}(Y'_n, \emptyset, \emptyset) \end{aligned} \quad (46)$$

where  $S_{\vec{\alpha}}^{-1}$  is the inverse Shapovalov matrix,  $\Gamma_{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3}$  are the 3-point functions. Here we rewrite the positions of punctures as

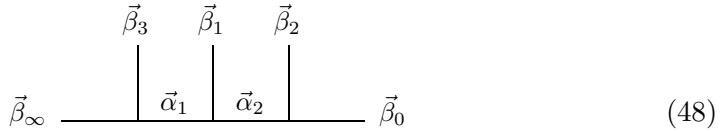
$$z_k = q_1 \cdots q_k \quad (k = 1, \dots, n), \quad (47)$$

since in the  $SU(2)$  case of original AGT relation, these  $q_k$ 's are identified with the coupling constants  $\tau_k$  of quiver gauge theory as  $q_k = e^{2\pi i \tau_k}$ .

Now we calculate the 5-point correlation functions, and check whether or not the AGT-W relation are satisfied. According to the AGT-W conjecture, these functions should correspond to the partition functions of the  $SU(3) \times SU(3)$  or  $SU(3) \times SU(2)$  quiver gauge theory.

### 4.1 $SU(3) \times SU(3)$ quiver

In this subsection, we discuss the  $SU(3) \times SU(3)$  case. The corresponding 5-point correlation function is that of the following diagram:



The momenta  $\vec{\alpha}$ 's and  $\vec{\beta}$ 's are of the form:

$$\begin{aligned} \vec{\alpha}_j &= Q\vec{\rho} + i\vec{\alpha}'_j \quad (j = 1, 2), \quad \vec{\beta}_k = Q\vec{\rho} + i\vec{\beta}'_k \quad (k = 0, \infty), \\ \vec{\beta}_1 &= (Q/2 + im_1)(-3\vec{\lambda}_3), \quad \vec{\beta}_2 = (Q/2 + im_2) \cdot 3\vec{\lambda}_1, \\ \vec{\beta}_3 &= (Q/2 + im_3)(-3\vec{\lambda}_3), \end{aligned} \quad (49)$$

where  $\sum_{p=1}^3 \alpha'_{j,p} = \sum_{p=1}^3 \beta'_{k,p} = 0$ , and all the parameters  $\vec{\alpha}_j$ ,  $\vec{\beta}_k$ ,  $m_i$  are real. Here we naturally set the momenta  $\vec{\beta}_2 \propto \vec{\lambda}_1$  and  $\vec{\beta}_3 \propto -\vec{\lambda}_3$ , since they correspond to the mass of fundamental and antifundamental fields in the gauge theory, respectively. On the other hand,  $\vec{\beta}_1$  corresponds to the mass of bifundamental field, so we don't have prescription for it at this point. It will be fixed later by requirement of AGT-W relation. Here we set  $\vec{\beta}_1 \propto -\vec{\lambda}_3$  in eq. (49). We note that it is also possible to set  $\vec{\beta}_1 \propto \vec{\lambda}_1$  for the following discussion without changing the result.

In the following, we calculate this 5-point correlation function, which can be written as the linear combination of

$$V_{(|Y_1|, |Y_2|)} := \sum_{|Y_1|=|Y'_1|} \sum_{|Y_2|=|Y'_2|} q_1^{|Y_1|} q_2^{|Y_2|} \Gamma'_{\vec{\beta}_\infty, \vec{\beta}_3, \vec{\alpha}_1}(\emptyset, \emptyset, Y_1) S_{\vec{\alpha}_1}^{-1}(Y_1, Y'_1) \\ \times \Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(Y'_1, \emptyset, Y_2) S_{\vec{\alpha}_2}^{-1}(Y_2, Y'_2) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(Y'_2, \emptyset, \emptyset), \quad (50)$$

according to eq. (46). Hereafter, the descendant level of  $V_{(|Y_1|, |Y_2|)}$  is denoted as  $[|Y_1|, |Y_2|]$ .

### 1-loop part

For 1-loop part, the internal states are also primary fields. Then the correlation function is given as

$$V_{(0,0)} = \Gamma'_{\vec{\beta}_\infty, \vec{\beta}_3, \vec{\alpha}_1}(\emptyset, \emptyset, \emptyset) \Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, \emptyset) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(\emptyset, \emptyset, \emptyset) \quad (51)$$

Using the explicit form of 3-point functions (23)–(25) and momenta (49), and the properties of Upsilon function (76)–(78), eq. (51) becomes

$$V_{(0,0)} = f(-\vec{\beta}_\infty) f(\vec{\beta}_0) \tilde{g}(m_1) g(m_2) \tilde{g}(m_3) \\ \times \prod_{l=1,2} \prod_{p < q} (\alpha'_{l,p} - \alpha'_{l,q})^2 \left| \Gamma_2(i\alpha'_{l,p} - i\alpha'_{l,q} + b) \Gamma_2(i\alpha'_{l,p} - i\alpha'_{l,q} + \frac{1}{b}) \right|^{-2} \\ \times \prod_{p,q=1}^3 \left| \Gamma_2\left(\frac{Q}{2} + i\alpha'_{1,p} - i\beta'_{\infty,q} + im_3\right) \right|^2 \left| \Gamma_2\left(\frac{Q}{2} + i\alpha'_{1,p} - i\alpha'_{2,q} - im_1\right) \right|^2 \\ \times \left| \Gamma_2\left(\frac{Q}{2} + i\alpha'_{2,p} - i\beta'_{0,q} + im_2\right) \right|^2, \quad (52)$$

up to the factors which only depend on  $b$ . Here we define

$$\begin{aligned} f(\vec{\beta}) &:= \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{-\vec{\beta} \cdot \vec{\rho}/b} \prod_{e>0} \Upsilon((Q\vec{\rho} - \vec{\alpha}) \cdot \vec{e}) \\ g(m) &:= \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{-3(Q/2+im)\vec{\lambda}_1 \cdot \vec{\rho}/b} \Upsilon(3(Q/2+im)) \\ \tilde{g}(m) &:= \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{-3(Q/2+im)\vec{\lambda}_3 \cdot \vec{\rho}/b} \Upsilon(-3(Q/2+im)). \end{aligned} \quad (53)$$

Now we show that  $V_{(0,0)}$  corresponds to the 1-loop part of the partition function (5) for  $SU(3) \times SU(3)$  gauge theory, when the parameters of the Toda theory and the gauge theory are identified appropriately. First, note that if we identify

$$b = \epsilon_1, \quad \frac{1}{b} = \epsilon_2, \quad i\bar{\alpha}'_l = \vec{a}_l : \text{VEV's of adjoint scalars}, \quad (54)$$

the second line of eq. (52) is equal to the product of  $\prod_l |z_{\text{vec}}^{\text{1lp}}(\vec{a}_l)|^2$  and the van der Monde factor.  $\prod_l \prod_{j < k} |\hat{a}_{l,j} - \hat{a}_{l,k}|^2$  This is the natural integral measure in the Coulomb branch. Next, if we identify

$$\bar{\mu}_{\bar{p}} = \frac{Q}{2} + im_3 - i\beta'_{\infty, \bar{p}}, \quad m = \frac{Q}{2} + im_1, \quad \mu_p = \frac{Q}{2} - im_2 + i\beta'_{0,p}, \quad (55)$$

for the mass of antifundamental, bifundamental and fundamental fields, the factors in the third and fourth lines of (52) are equal to  $\prod_{\bar{p}} |z_{\text{afd}}^{\text{1lp}}(\vec{a}_1, \bar{\mu}_{\bar{p}})|^2$ ,  $|z_{\text{bfd}}^{\text{1lp}}(\vec{a}_1, \vec{a}_2, m)|^2$  and  $\prod_p |z_{\text{fd}}^{\text{1lp}}(\vec{a}_2, \mu_p)|^2$ , respectively.

To summarize, with the identification of parameters (54)–(55), the correlation function at this level can be written as, up to some factors,

$$V_{(0,0)} = |Z_{\text{1-loop}}|^2 \quad (56)$$

where  $Z_{\text{1-loop}}$  is the 1-loop factor of Nekrasov partition function for  $SU(3) \times SU(3)$  quiver gauge theory. This is the AGT-W relation for the 1-loop part.

### Instanton part

For instanton part, we check the AGT-W relation for each instanton level  $[(Y_1), (Y_2)]$  in eq. (50).

### Level $[n, 0]$ or $[0, n]$

For these levels, the discussion is almost parallel to 4-point function's case, which corresponds to AGT-W relation for  $SU(3)$  gauge theory [44], since one of the intermediate operators is set to be primary. For simple examples, the partition functions for  $SU(3)_1 \times SU(3)_2$  quiver gauge theory at the instanton level  $[1,0]$  and  $[0,1]$  are

$$\begin{aligned} Z_{\text{inst}}^{([1],\emptyset)} &= q_1 \sum_{i=1}^3 \frac{\prod_{\bar{p}=1}^3 (\hat{a}_{1,i} + \bar{\mu}_{\bar{p}}) \prod_{j=1}^3 (\hat{a}_{1,i} - \hat{a}_{2,j} - m)}{\prod_{i \neq k} (\hat{a}_{1,i} - \hat{a}_{1,k})(\hat{a}_{1,i} - \hat{a}_{1,k} + \epsilon_+)} \\ &=: \sum_{i=1}^3 R_{1,i}(\hat{a}_{1,i}), \end{aligned} \quad (57)$$

$$\begin{aligned} Z_{\text{inst}}^{(\emptyset,[1])} &= q_2 \sum_{i=1}^3 \frac{\prod_{j=1}^3 (\hat{a}_{1,j} - \hat{a}_{2,i} - m + \epsilon_+) \prod_{p=1}^3 (\hat{a}_{2,i} - \mu_p + \epsilon_+)}{\prod_{i \neq k} (\hat{a}_{2,i} - \hat{a}_{2,k})(\hat{a}_{2,i} - \hat{a}_{2,k} + \epsilon_+)} \\ &=: \sum_{i=1}^3 R_{2,i}(\hat{a}_{2,i}), \end{aligned} \quad (58)$$

where  $\epsilon_+ = \epsilon_1 + \epsilon_2$  is identified with  $Q$  from eq. (54). Therefore, eq. (57) is the same form as the level-1 instanton partition function for  $SU(3)_1$  gauge theory with the mass of fundamental fields  $\mu_j = Q/2 - im_1 + i\hat{a}_{2,j}$ . Similarly, eq. (58) corresponds to that for  $SU(3)_2$  theory with the mass of antifundamental fields  $\bar{\mu}_j = Q/2 + im_1 - i\hat{a}_{1,j}$ . Compared with eq. (55), they are exactly the relevant 4-point functions which is obtained by cutting one of internal lines  $\vec{\alpha}_2/\vec{\alpha}_1$  in our 5-point function (48) and regarding it as an external line. It has been already shown that these 4-point functions correspond to some conformal blocks of  $A_2$  Toda theory [44].

Now we also have to mention the  $U(1)$ -factors [4]. In the setup of AGT-W relation, strictly speaking, we discuss  $U(3) \times U(3)$  quiver gauge theory. The  $U(1)$  part of gauge symmetry is actually decoupled, but the  $U(1)$  flavor symmetry remains, which causes additional  $U(1)$ -factors. Therefore, unless we multiply it by Nekrasov partition function for  $SU(3) \times SU(3)$ , we cannot establish the AGT-W relation. In this case, the  $U(1)$ -factor is of the form

$$Z_{U(1)} = (1 - q_1)^{\nu_1} (1 - q_2)^{\nu_2} (1 - q_1 q_2)^{\nu_3}, \quad (59)$$

where

$$\begin{aligned}\nu_1 &= -3 \left( \frac{Q}{2} + im_3 \right) \left( \frac{Q}{2} - im_1 \right), \quad \nu_2 = -3 \left( \frac{Q}{2} + im_1 \right) \left( \frac{Q}{2} + im_2 \right), \\ \nu_3 &= -3 \left( \frac{Q}{2} + im_3 \right) \left( \frac{Q}{2} + im_2 \right).\end{aligned}\quad (60)$$

On the other hand, the correlation function of  $A_2$  Toda theory at these levels are

$$\begin{aligned}V_{(1,0)} &= \sum_{Y_1, Y'_1} \frac{\Gamma'_{\vec{\beta}_\infty, \vec{\beta}_3, \vec{\alpha}_1}(\emptyset, \emptyset, Y_1) S_{\vec{\alpha}_1}^{-1}(Y_1, Y'_1) \Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(Y'_1, \emptyset, \emptyset)}{\Gamma_{\vec{\beta}_\infty, \vec{\beta}_3, \vec{\alpha}_1}(\emptyset, \emptyset, \emptyset) \Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, \emptyset)} V_{(0,0)}, \\ V_{(0,1)} &= \sum_{Y_2, Y'_2} \frac{\Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, Y_2) S_{\vec{\alpha}_2}^{-1}(Y_2, Y'_2) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(Y'_2, \emptyset, \emptyset)}{\Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, \emptyset) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(\emptyset, \emptyset, \emptyset)} V_{(0,0)},\end{aligned}\quad (61)$$

where  $Y_1, Y'_1, Y_2, Y'_2 \in \{([1], \emptyset), (\emptyset, [1])\}$ . After everything is put together, we obtain the expected result, that is,

$$\frac{V_{(1,0)}}{V_{(0,0)}} = Z_{\text{inst}}^{([1], \emptyset)} + \nu_1, \quad \frac{V_{(0,1)}}{V_{(0,0)}} = Z_{\text{inst}}^{(\emptyset, [1])} + \nu_2. \quad (62)$$

It is straightforward to discuss the similar relations at level  $[n, 0]$  or  $[0, n]$  ( $n > 1$ ) by computer. In fact, we have checked them for  $n = 2, 3$ .

### Level $[n_1, n_2]$ with $n_1, n_2 > 0$

From above discussion, we identified all the parameters of 4-dim  $SU(3)$  quiver gauge theory and 2-dim  $A_2$  Toda theory, and obtained all necessary relations between them, *i.e.* eqs. (54), (55) and (60), for checking AGT-W relation. With these identification of parameters, we carry out the check of the relation for the instanton level  $[n_1, n_2]$  ( $n_1, n_2 > 0$ ).

The simplest example is level  $[1, 1]$ . The partition function and  $U(1)$  factor of gauge theory at this level are

$$\begin{aligned}Z_{\text{inst}}^{([1], [1])} &= \sum_{i,j=1}^3 R_{1,i}(\hat{a}_{1,i}) R_{2,j}(\hat{a}_{2,j}) \frac{(\hat{a}_{1,i} - \hat{a}_{2,j} - m + \epsilon_1)(\hat{a}_{1,i} - \hat{a}_{2,j} - m + \epsilon_2)}{(\hat{a}_{1,i} - \hat{a}_{2,j} - m + \epsilon_+)(\hat{a}_{1,i} - \hat{a}_{2,j} - m)}, \\ Z_{U(1)}^{[1,1]} &= \nu_1 Z_{\text{inst}}^{(0,[1])} + \nu_2 Z_{\text{inst}}^{([1],0)} + \nu_1 \nu_2 + \nu_3.\end{aligned}\quad (63)$$

On the other hand, the correlation function of Toda theory at this level is

$$V_{(1,1)} = \sum_{Y_1, Y'_1, Y_2, Y'_2} \Gamma_{\vec{\beta}_\infty, \vec{\beta}_3, \vec{\alpha}_1}(\emptyset, \emptyset, Y_1) S_{\vec{\alpha}_1}^{-1}(Y_1, Y'_1) \Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(Y'_1, \emptyset, Y_2) \\ \times S_{\vec{\alpha}_2}^{-1}(Y_2, Y'_2) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(Y'_2, \emptyset, \emptyset), \quad (64)$$

where  $Y_1, Y'_1, Y_2, Y'_2 \in \{([1], \emptyset), (\emptyset, [1])\}$ . Then we can obtain the expected result, that is,

$$\frac{V_{(1,1)}}{V_{(0,0)}} = Z_{\text{inst}}^{([1],[1])} + Z_{U(1)}^{[1,1]}. \quad (65)$$

We have also successfully checked the  $(n_1, n_2) = (1, 2), (2, 1)$  cases by computer. Since the formula become complicated, we don't write their explicit form here.

### Summary

We check the AGT-W relation in  $SU(3) \times SU(3)$  quiver case for the 1-loop part and the instanton corrections for the level  $[|Y_1|, |Y_2|]$  with  $|Y_1| + |Y_2| \leq 3$ . While this is only partial result toward the proof, the coincidence of the explicit formula in both side is quite nontrivial and very convincing. Its generalization to the case of  $SU(3) \times \cdots \times SU(3)$  quiver seems straightforward. Moreover, for 1-loop part, it is easy to generalize to the case of  $SU(N) \times \cdots \times SU(N)$  quiver, where the argument is a straightforward generalization of the  $SU(3)$  quiver case.

## 4.2 $SU(3) \times SU(2)$ quiver

As we mentioned before, the other kind of 5-point correlation function of  $A_2$  Toda theory should correspond to the partition function of  $SU(3) \times SU(2)$  quiver gauge theory. Let us now discuss the AGT-W relation in this case.

### 1-loop part

The puncture  $V_{\vec{\beta}_0}(0)$  now becomes the ‘simple’ puncture, instead of ‘full’ puncture. So we must consider the correlation function of the diagram (48) with  $\vec{\beta}_0 = (Q/2 + im_0) \cdot 3\vec{\lambda}_1$ .

After having set this value, we meet an immediate problem. To see this, the last factor in eq. (50) contains two vertices  $\beta_0, \beta_2$  to be proportional to

$\vec{\lambda}_1$ . This  $\Gamma$  factor, at the level  $|Y'_2| = 0$ , is written by the formula (24) where one of the vector, say  $\vec{\alpha}_2$ , is proportional to  $\vec{\lambda}_1$ .<sup>2</sup> Then in the numerator, we have a factor  $\Upsilon((Q\vec{\rho} - \vec{\alpha}_2) \cdot \vec{e}_2) = \Upsilon(Q)$ , which vanishes because of the property of  $\Upsilon$  function. Such zero factors always exist for the linear quiver whose product gauge groups have different ranks. While it may imply a limitation to AGT-W conjecture, in the following study, we drop such factors since they don't depend on the momentum of the intermediate operator. In appendix D, we derive some properties of 3-point functions where two of the three operators have level-1 singular state, especially the constraint for the third operator to have non-vanishing 3-point function. As it is explained there, from the conformal Ward identities, we need a relation between  $\Delta$  and  $w$  for the third generator. This is, however, a much weaker condition than eqs. (24)–(25), where the correlation function vanishes for arbitrary vertex. This may imply that we need some modifications in such special cases. In this paper, however, we will not try to go further in this direction, but simply drop the zero factor and study the consequence. This may be justified, since it does not depend on the parameters of the theory.

For the internal momentum  $\vec{\alpha}_2$  in eq. (48), we must set it as a 1-parameter vector, since it will correspond to the VEV of  $SU(2)$  gauge group. There is arbitrariness to choose the form of  $\vec{\alpha}_2$ , but a natural choice will be  $\vec{\alpha}_2 = Q\vec{\rho} + i(\alpha'_2, -\alpha'_2, 0)$  or  $Q\vec{\rho} + i(\alpha'_2, \alpha'_2, -2\alpha'_2)$ . If we adopt the later one, van der Monde factor becomes 0. This is not desirable, so we use the former one. We expect that  $V_{(0,0)}$  corresponds to the 1-loop part of the partition function for  $SU(3) \times SU(2)$  quiver gauge theory. In order to achieve this, however, we are forced to impose an additional condition  $m_2 + m_0 = 0$ . It helps to cancel out unnecessary factors of  $V_{(0,0)}$  and we obtain

$$\begin{aligned} V_{(0,0)} &= \prod_{p < q}^3 (\alpha'_{1,p} - \alpha'_{1,q})^2 \left| \Gamma_2(i\alpha'_{1,p} - i\alpha'_{1,q} + b) \Gamma_2(i\alpha'_{1,p} - i\alpha'_{1,q} + \frac{1}{b}) \right|^{-2} \\ &\quad \times 4\alpha'^2_2 \left| \Gamma_2(2i\alpha'_2 + b) \Gamma_2(2i\alpha'_2 + \frac{1}{b}) \right|^{-2} \\ &\quad \times \prod_{p,q=1}^3 \left| \Gamma_2\left(\frac{Q}{2} + i\alpha'_{1,p} - i\beta'_{\infty,q} + im_3\right) \right|^2 \prod_{p=1}^3 \left| \Gamma_2\left(\frac{Q}{2} + i\alpha'_{1,p} + im_1\right) \right|^2 \end{aligned}$$

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<sup>2</sup> Here, please don't confuse this  $\vec{\alpha}_2$  in eq. (24) with that in eq. (48).

$$\begin{aligned} & \times \prod_{p=1}^3 \prod_{q=1}^2 \left| \Gamma_2 \left( \frac{Q}{2} + i\alpha'_{1,p} - i\alpha'_{2,q} - im_1 \right) \right|^2 \cdot \\ & \cdot \left| \Gamma_2 \left( \frac{Q}{2} - i\alpha'_2 - 3im_2 \right) \Gamma_2 \left( \frac{Q}{2} + i\alpha'_2 - 3im_2 \right) \right|^2 \end{aligned} \quad (66)$$

up to some constant which is independent of  $\vec{\alpha}_1$  and  $\vec{\alpha}_2$ . As in the previous subsection, we can see that if we identify

$$\begin{aligned} i\vec{\alpha}'_1 &= \vec{a}_1 : \text{VEV's of } SU(3) \text{ adjoint scalar} \\ (i\alpha'_2, -i\alpha'_2) &= \vec{a}_2 : \text{VEV's of } SU(2) \text{ adjoint scalar}, \end{aligned} \quad (67)$$

the first line of eq. (66) is equal to the product of  $|z_{\text{vec}}^{\text{1lp}}(\vec{a}_1)|^2$  and van der Monde factor  $\prod_{j < k} |\hat{a}_{1,j} - \hat{a}_{1,k}|^2$  for  $SU(3)$  gauge group and the second line is equal to that for  $SU(2)$  gauge group. Also, we can find that if we identify

$$\bar{\mu}_{\bar{p}} = \frac{Q}{2} + im_3 - i\beta'_{\infty, \bar{p}}, \quad \nu = \frac{Q}{2} + im_1, \quad m = \frac{Q}{2} + im_1, \quad \mu = \frac{Q}{2} + 3im_2, \quad (68)$$

with the mass of three  $SU(3)$  antifundamental, a  $SU(3)$  fundamental, a  $SU(3) \times SU(2)$  bifundamental and a  $SU(2)$  fundamental fields, the factors in the third and fourth lines of eq. (66) equal to  $\prod_{\bar{p}} |z_{\text{afd}}^{\text{1lp}}(\vec{a}_1, \bar{\mu}_{\bar{p}})|^2$ ,  $|z_{\text{fd}}^{\text{1lp}}(\vec{a}_1, \nu)|^2$ ,  $|z_{\text{bfd}}^{\text{1lp}}(\vec{a}_1, \vec{a}_2, m)|^2$  and  $|z_{\text{fd}}^{\text{1lp}}(\vec{a}_2, \mu)|^2$ , respectively.

Therefore, we see that eq. (66) can be written as  $V_{(0,0)} = |Z_{\text{1-loop}}|^2$ , where  $Z_{\text{1-loop}}$  is the 1-loop factor of Nekrasov partition function for  $SU(3) \times SU(2)$  quiver gauge theory, up to the zero factors which we mentioned above. Here we have to note, however, we have met additional problems in the case of  $SU(3) \times SU(2)$  quiver. First,  $SU(3)$  fundamental field and  $SU(3) \times SU(2)$  bifundamental field have the same mass  $\nu = m = Q/2 + im_1$ . In the gauge theory side, it is not necessary for the two fields to have the same mass. How can we make these two mass independent in Toda theory? Secondly, we need to impose the condition  $m_2 + m_0 = 0$  to get the correspondence. This is artificial and there seems no physical meaning in Toda theory. We don't know a correct answer yet, but one possibility to resolve this puzzle may be that the form of the 3-point functions (24)–(25) could be modified, when more than one of momenta  $\vec{\alpha}_1$  or/and  $\vec{\alpha}_2$  are also proportional to  $\vec{\lambda}_1$  or  $\vec{\lambda}_3$ . The representation of W-algebra for a degenerate field is very different from that for a non-degenerate one. Therefore, it may be possible that the form of the 3-point functions changes when two vertex operators are degenerate fields, which may also resolve the problem of zero factor.

### Instanton part

For the instanton level  $[n, 0]$ , we can discuss in a similar way to the  $SU(3) \times SU(3)$  case. In the simplest  $n = 1$  case, the partition function is

$$Z_{\text{inst}}^{([1], \emptyset)} = q_1 \sum_{i=1}^3 \frac{\prod_{\bar{p}=1}^3 (\hat{a}_{1,i} + \bar{\mu}_{\bar{p}}) \prod_{j=1}^2 (\hat{a}_{1,i} - \hat{a}_{2,j} - m) \cdot (\hat{a}_{1,i} - \nu + \epsilon_+)}{\prod_{i \neq k} (\hat{a}_{1,i} - \hat{a}_{1,k}) (\hat{a}_{1,i} - \hat{a}_{1,k} + \epsilon_+)} , \quad (69)$$

which is the same form as the level-1 instanton partition function for  $SU(3)$  gauge theory with the mass of fundamental fields  $\nu$  and  $\mu_j = Q/2 - im_1 + i\hat{a}_{2,j}$ . Therefore, together with  $U(1)$ -factor (60), we successfully obtain

$$\frac{V_{(1,0)}}{V_{(0,0)}} = Z_{\text{inst}}^{([1], \emptyset)} + \nu_1 . \quad (70)$$

We also have checked the correspondence in the  $n = 2$  case by computer.

On the other hand, however, for the level  $[0, n]$ , the correspondence is rather nontrivial. The partition function in the  $n = 1$  case is

$$Z_{\text{inst}}^{(\emptyset, [1])} = q_2 \sum_{i=1}^2 \frac{\prod_{j=1}^3 (\hat{a}_{1,j} - \hat{a}_{2,i} - m + \epsilon_+) \cdot (\hat{a}_{2,i} - \mu + \epsilon_+)}{\prod_{i \neq k} (\hat{a}_{2,i} - \hat{a}_{2,k}) (\hat{a}_{2,i} - \hat{a}_{2,k} + \epsilon_+)} , \quad (71)$$

which is the same form as the level-1 instanton partition function for  $SU(2)$  gauge theory. Therefore, in order to match this function with Toda correlation function

$$V_{(0,1)} = \sum_{Y_2, Y'_2} \frac{\Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, Y_2) S_{\vec{\alpha}_2}^{-1}(Y_2, Y'_2) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(Y'_2, \emptyset, \emptyset)}{\Gamma_{\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2}(\emptyset, \emptyset, \emptyset) \Gamma_{\vec{\alpha}_2, \vec{\beta}_2, \vec{\beta}_0}(\emptyset, \emptyset, \emptyset)} V_{(0,0)} , \quad (72)$$

we need to sum up only the  $L_{-n}$  descendants of  $V_{\vec{\alpha}_2}$ , *i.e.*  $Y_2, Y'_2 \in ([1], \emptyset)$ . This is very strange situation, since it means that  $W_{-n}$  generators cannot live in some region of Seiberg-Witten curve. This region is identified as the region which a D6-brane passes through in the intersecting D4/NS5-branes' system, when the present  $SU(3) \times SU(2)$  system is realized from the  $SU(3) \times SU(3)$  system by moving a D6-brane from infinite distance. Then if we ignore the  $W_{-n}$  descendants here, we can obtain the desirable result

$$\frac{V_{(0,1)}}{V_{(0,0)}} = Z_{\text{inst}}^{(\emptyset, [1])} + \nu_2 . \quad (73)$$

We also have checked the  $n = 2$  case by computer. However, at this moment, we have no persuasive reason to justify this procedure in Toda theory.

Finally, we mention the case of the level  $[n_1, n_2]$  with  $n_1, n_2 > 0$ . In this case, we also need to ignore the  $W_{-n}$  descendants of  $V_{\tilde{\alpha}_2}$ . We have checked it in the  $[1, 1]$  case.

### Summary

We manage to check the AGT-W relation for  $SU(3) \times SU(2)$  quiver gauge theory in the 1-loop part and the instanton part at the level  $[[Y_1], [Y_2]]$  with  $|Y_1| + |Y_2| \leq 2$ . However, the correspondence is quite nontrivial, since there remain the mysteries of the vanishing factor  $\Upsilon(Q)$ , the strange conditions on mass and the ignorance of  $W_{-n}$  descendants. The investigation of these problems must be an important future work.

## 5 Conclusion

In this paper, we study and confirm the AGT-W relation for  $SU(3) \times SU(3)$  gauge theory in the 1-loop factor and also the lower level factors in the instanton part by solving conformal Ward identity by computer. Our calculation method is valid for the general  $SU(3) \times \cdots \times SU(3)$  cases, and the generalization to these cases is straightforward. Moreover, such relation seems to hold for the linear  $SU(N) \times \cdots \times SU(N)$  quiver gauge theories as well, at least for the 1-loop part.

For the  $SU(3) \times SU(2)$  quiver, however, we encountered some problems which include the undesirable vanishing factor, the extra conditions on the mass for bifundamental and fundamental matter fields, and we are forced to ignore  $W_{-n}$  descendants in some part of the correlation function. For these cases, we need to have some modification to AGT-W relation and/or our calculation method by the decomposing Gaiotto curve. After these problems are solved, we will be able to straightforwardly confirm the general  $SU(3) \times \cdots \times SU(3) \times SU(2)$  and  $SU(2) \times SU(3) \times \cdots \times SU(3) \times SU(2)$  case, that is, all the case of AGT-W relation for linear  $SU(3)$  quivers. Therefore, we hope to clarify them in the future paper.

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## A Properties of double Gamma function $\Gamma_2$ and Upsilon function $\Upsilon$

The double Gamma function  $\Gamma_2(x|\epsilon_1, \epsilon_2)$  is defined as

$$\Gamma_2(x|\epsilon_1, \epsilon_2) = \exp \frac{d}{ds} \Big|_{s=0} \zeta_2(s; x|\epsilon_1, \epsilon_2), \quad (74)$$

where

$$\zeta_2(s; x|\epsilon_1, \epsilon_2) := \sum_{m,n} (m\epsilon_1 + n\epsilon_2 + x)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1} e^{-tx}}{(1 - e^{-\epsilon_1 t})(1 - e^{\epsilon_2 t})}, \quad (75)$$

where  $\Gamma(s)$  is ordinary Gamma function. This function  $\Gamma_2(x|\epsilon_1, \epsilon_2)$  is often written as  $\Gamma_2(x)$  if there is no confusion, and satisfies the following relations:

$$\Gamma_2(x)^* = \Gamma_2(x^*) \quad (76)$$

$$\Gamma_2(x + \epsilon_1)\Gamma_2(x + \epsilon_2) = x\Gamma_2(x)\Gamma_2(x + \epsilon_1 + \epsilon_2). \quad (77)$$

The Upsilon function  $\Upsilon(x)$  can be written as the product of double Gamma functions:

$$\Upsilon(x) = \frac{1}{\Gamma_2(x)\Gamma_2(Q-x)}, \quad (78)$$

which is also very important for AGT and AGT-W relation.

## B Shapovalov matrix

As we mentioned in the main text, Shapovalov matrix is given as eq. (28). The concrete forms of its elements up to level-2 descendants are given in eq. (65) of [44], for example. In this paper, we also use the concrete forms

of level-3 descendants, so we show them in this appendix. In the following, we denote these elements (28) as  $S(\mathcal{L}_{-Y_1}, \mathcal{L}_{-Y_2})$ .

$$\begin{aligned}
S(L_{-3}, L_{-3}) &= 6\Delta + 2c, & S(L_{-3}, L_{-2}L_{-1}) &= 10\Delta, & S(L_{-3}, L_{-1}^3) &= 24\Delta, \\
S(L_{-3}, L_{-2}W_{-1}) &= 15w, & S(L_{-3}, L_{-1}^2W_{-1}) &= 24w, & S(L_{-3}, L_{-1}W_{-2}) &= 36w, \\
S(L_{-3}, L_{-1}W_{-1}^2) &= 90D\Delta, & S(L_{-3}, W_{-3}) &= 9w, & S(L_{-3}, W_{-2}W_{-1}) &= 36D\Delta, \\
S(L_{-3}, W_{-1}^3) &= \frac{189}{2}w(3\Delta + 1), \\
S(L_{-2}L_{-1}, L_{-2}L_{-1}) &= \Delta(8(\Delta + 1) + c), & S(L_{-2}L_{-1}, L_{-1}^3) &= 12\Delta(3\Delta + 1), \\
S(L_{-2}L_{-1}, L_{-2}W_{-1}) &= \frac{3}{2}w(8(\Delta + 1) + c), & S(L_{-2}L_{-1}, L_{-1}^2W_{-1}) &= 18w(3\Delta + 1), \\
S(L_{-2}L_{-1}, L_{-1}W_{-2}) &= 12w(\Delta + 3), & S(L_{-2}L_{-1}, L_{-1}W_{-1}^2) &= 9(D\Delta(5\Delta + 9) + 6w^2), \\
S(L_{-2}L_{-1}, W_{-3}) &= 21w, & S(L_{-2}L_{-1}, W_{-2}W_{-1}) &= 18(3D\Delta + w^2), \\
S(L_{-2}L_{-1}, W_{-1}^3) &= \frac{135w}{2}(D(3\Delta + 1) + \kappa(5\Delta + 1)), \\
S(L_{-1}^3, L_{-1}^3) &= 24\Delta(\Delta + 1)(2\Delta + 1), & S(L_{-1}^3, L_{-2}W_{-1}) &= 18w(3\Delta + 1), \\
S(L_{-1}^3, L_{-1}^2W_{-1}) &= 36w(\Delta + 1)(2\Delta + 1), & S(L_{-1}^3, L_{-1}W_{-2}) &= 72w(\Delta + 1), \\
S(L_{-1}^3, L_{-1}W_{-1}^2) &= 54(\Delta + 1)(3D\Delta + 2w^2), & S(L_{-1}^3, W_{-3}) &= 60w, \\
S(L_{-1}^3, W_{-2}W_{-1}) &= 108(D\Delta + w^2), & S(L_{-1}^3, W_{-1}^3) &= 81w(3D(3\Delta + 4) + 2w^2 + 1), \\
S(L_{-2}W_{-1}, L_{-2}W_{-1}) &= \frac{9}{4}D\Delta(8(\Delta + 1) + c), & S(L_{-2}W_{-1}, L_{-1}^2W_{-1}) &= 27(D\Delta(\Delta + 1) + 2w^2), \\
S(L_{-2}W_{-1}, L_{-1}W_{-2}) &= 18(3D\Delta + w^2), & S(L_{-2}W_{-1}, L_{-1}W_{-1}^2) &= \frac{27}{2}Dw(11\Delta + 9), \\
S(L_{-2}W_{-1}, W_{-3}) &= \frac{63}{2}D\Delta, & S(L_{-2}W_{-1}, W_{-2}W_{-1}) &= 27Dw(\Delta + 3), \\
S(L_{-2}W_{-1}, W_{-1}^3) &= \frac{405}{4}[D^2\Delta(3\Delta + 1) + \kappa(D\Delta(\Delta + 1) + 4w^2)], \\
S(L_{-1}^2W_{-1}, L_{-1}^2W_{-1}) &= 18(\Delta + 1)(D\Delta(2\Delta + 3) + 4w^2), \\
S(L_{-1}^2W_{-1}, L_{-1}W_{-2}) &= 36(D\Delta(2\Delta + 3) + w^2), \\
S(L_{-1}^2W_{-1}, L_{-1}W_{-1}^2) &= 27w(D(\Delta + 3)(4\Delta + 3) + 2w^2), \\
S(L_{-1}^2W_{-1}, W_{-3}) &= 90D\Delta, & S(L_{-1}^2W_{-1}, W_{-2}W_{-1}) &= 162wD(\Delta + 1), \\
S(L_{-1}^2W_{-1}, W_{-1}^3) &= \frac{243}{2}[D^2\Delta(3\Delta + 7) + 2D(\Delta + 3)w^2 + \kappa(D\Delta(\Delta + 1) + 4w^2)], \\
S(L_{-1}W_{-2}, L_{-1}W_{-2}) &= \frac{9\kappa}{4}\Delta(8\Delta^2 + (54 + c)\Delta + 14 + c), \\
S(L_{-1}W_{-2}, L_{-1}W_{-1}^2) &= \frac{27\kappa}{16}(112\Delta^2 + (210 - c)\Delta + 34 - c), & S(L_{-1}W_{-2}, W_{-3}) &= 45\Delta(D + 1), \\
S(L_{-1}W_{-2}, W_{-2}W_{-1}) &= \frac{27\kappa}{8}w(8\Delta^2 + (54 + c)\Delta + 14 + c), \\
S(L_{-1}W_{-2}, W_{-1}^3) &= \frac{243\kappa}{32}[w^2(48\Delta + 46 + c) + D\Delta(128\Delta + 62 + c)], \\
S(L_{-1}W_{-1}^2, L_{-1}W_{-1}^2) &= \frac{81}{2}[D^2\Delta(2\Delta^2 + 5\Delta + 11) + \kappa D\Delta(\Delta + 1)(\Delta + 2) + 4w^2(D(\Delta + 3) + \kappa(\Delta + 2))], \\
S(L_{-1}W_{-1}^2, W_{-3}) &= \frac{135}{2}w(3D + 1), \\
S(L_{-1}W_{-1}^2, W_{-2}W_{-1}) &= 81D\Delta(D(2\Delta + 1) + \kappa(\Delta + 1)) + \frac{81\kappa}{32}w^2(48\Delta + 142 + c), \\
S(L_{-1}W_{-1}^2, W_{-1}^3) &= \frac{729}{4}w[D^2(\Delta + 1)(2\Delta + 5) + \kappa(D(\Delta^2 + 18\Delta + 5) + 4w^2)],
\end{aligned}$$

$$S(W_{-3}, W_{-3}) = \frac{27}{2}D\Delta + \frac{3}{2}(24\Delta + c), \quad S(W_{-3}, W_{-2}W_{-1}) = \frac{27\kappa}{2}w(5\Delta + 1) + 54w,$$

$$S(W_{-3}, W_{-1}^3) = \frac{81\kappa}{64}[128w^2 + D\Delta(320\Delta + 302 + 25c)],$$

$$S(W_{-2}W_{-1}, W_{-2}W_{-1}) = \frac{81\kappa}{16}(32w^2 + D\Delta(\Delta + 1)(8\Delta + 14 + c)),$$

$$S(W_{-2}W_{-1}, W_{-1}^3) = \frac{729\kappa}{64}Dw(48\Delta^2 + (174 + c)\Delta + 62 + c),$$

$$S(W_{-1}^3, W_{-1}^3) = \frac{729}{8}[3D^3\Delta(\Delta + 1)(2\Delta + 1) + \kappa D^2\Delta(9\Delta^2 + 69\Delta + 44) + \kappa^2 D\Delta(\Delta + 1)(3\Delta + 5)] \\ + \frac{729\kappa}{8}w^2[4D(-5D + \kappa(5\Delta + 1))(9\Delta + 16) + 8(4\Delta + 3)],$$

where  $D := \kappa(\Delta + \frac{1}{5}) - \frac{1}{5}$ ,  $\kappa := \frac{32}{22+5c}$ .

## C 3-point functions

In §3, we show the general procedure to calculate 3-point functions (29), which we must obtain in order to check the AGT-W relation. In this appendix, we show the concrete forms of  $\Gamma(\mathcal{L}_{-Y_\infty}, \mathcal{L}_{-Y_0}) := \Gamma_{\vec{\alpha}_\infty, \vec{\alpha}_1, \vec{\alpha}_0}(Y_\infty, \emptyset, Y_0)$  at the descendant level  $[|Y_\infty|, |Y_0|]$  with  $|Y_\infty| + |Y_0| \leq 3$ . Those of level [1,0], [0,1], [2,0], [0,2] are already given in [44].

### Level [1,0] and [0,1]

$$\Gamma(L_{-1}, \emptyset) = \Delta_\infty + \Delta_1 - \Delta_0, \quad \Gamma(W_{-1}, \emptyset) = w_\infty - w_1 - w_0 + \chi\Gamma(L_{-1}, \emptyset),$$

$$\Gamma(\emptyset, L_{-1}) = -\Delta_\infty + \Delta_1 + \Delta_0, \quad \Gamma(\emptyset, W_{-1}) = -w_\infty + w_1 + w_0 - \chi\Gamma(\emptyset, L_{-1}),$$

where  $\chi := 3w_1/2\Delta_1$ .

### Level [2,0]

$$\Gamma(L_{-2}, \emptyset) = \Gamma(L_{-1}, \emptyset) + \Delta_1, \quad \Gamma(L_{-1}^2, \emptyset) = [\Gamma(L_{-1}, \emptyset) + 1]\Gamma(L_{-1}, \emptyset),$$

$$\Gamma(L_{-1}W_{-1}, \emptyset) = [\Gamma(L_{-1}, \emptyset) + 1]\Gamma(W_{-1}, \emptyset), \quad \Gamma(W_{-2}, \emptyset) = \Gamma(W_{-1}, \emptyset) + \chi\Gamma(L_{-1}, \emptyset),$$

$$\Gamma(W_{-1}^2, \emptyset) = [\Gamma(W_{-1}, \emptyset) + \chi]\Gamma(W_{-1}, \emptyset) + \frac{9}{2}D_\infty\Gamma(L_{-1}, \emptyset),$$

where  $D_\infty := \kappa(\Delta_\infty + \frac{1}{5}) - \frac{1}{5}$ .

### Level [1,1]

$$\Gamma(L_{-1}, L_{-1}) = \Gamma(L_{-1}, \emptyset)\Gamma(\emptyset, L_{-1}) + (\Delta_\infty - \Delta_1 + \Delta_0),$$

$$\Gamma(L_{-1}, W_{-1}) = [\Gamma(L_{-1}, \emptyset) - 1]\Gamma(\emptyset, W_{-1}) + 3w_0,$$

$$\Gamma(W_{-1}, L_{-1}) = \Gamma(W_{-1}, \emptyset)[\Gamma(\emptyset, L_{-1}) - 1] + 3w_\infty,$$

$$\Gamma(W_{-1}, W_{-1}) = [\Gamma(W_{-1}, \emptyset) - \chi]\Gamma(\emptyset, W_{-1}) + \frac{9}{2}D_0(\Delta_\infty - \Delta_1),$$

where  $D_0 := \kappa(\Delta_0 + \frac{1}{5}) - \frac{1}{5}$ .

Level [0,2]

$$\begin{aligned}\Gamma(\emptyset, L_{-2}) &= \Gamma(\emptyset, L_{-1}) + \Delta_1, \quad \Gamma(\emptyset, L_{-1}^2) = [\Gamma(\emptyset, L_{-1}) + 1]\Gamma(\emptyset, L_{-1}), \\ \Gamma(\emptyset, L_{-1}W_{-1}) &= [\Gamma(\emptyset, L_{-1}) + 1]\Gamma(\emptyset, W_{-1}), \quad \Gamma(\emptyset, W_{-2}) = \Gamma(\emptyset, W_{-1}) - \chi\Gamma(\emptyset, L_{-1}), \\ \Gamma(\emptyset, W_{-1}^2) &= [\Gamma(\emptyset, W_{-1}) - \chi]\Gamma(\emptyset, W_{-1}) + \frac{9}{2}D_0\Gamma(\emptyset, L_{-1}).\end{aligned}$$

Level [3,0] We don't write the explicit forms of  $\Gamma(W_{-1}^3, \emptyset)$ , since it is very cumbersome.

$$\begin{aligned}\Gamma(L_{-3}, \emptyset) &= \Gamma(L_{-2}, \emptyset) + \Delta_1, \quad \Gamma(L_{-2}L_{-1}, \emptyset) = [\Gamma(L_{-2}, \emptyset) + 1]\Gamma(L_{-1}, \emptyset), \\ \Gamma(L_{-1}^3, \emptyset) &= [\Gamma(L_{-1}, \emptyset) + 2]\Gamma(L_{-1}^2, \emptyset), \quad \Gamma(L_{-2}W_{-1}, \emptyset) = [\Gamma(L_{-2}, \emptyset) + 1]\Gamma(W_{-1}, \emptyset), \\ \Gamma(L_{-1}^2W_{-1}, \emptyset) &= [\Gamma(L_{-1}, \emptyset) + 2]\Gamma(L_{-1}W_{-1}, \emptyset), \quad \Gamma(L_{-1}W_{-2}, \emptyset) = [\Gamma(L_{-1}, \emptyset) + 2]\Gamma(W_{-2}, \emptyset), \\ \Gamma(L_{-1}W_{-1}^2, \emptyset) &= [\Gamma(L_{-1}, \emptyset) + 2]\Gamma(W_{-1}^2, \emptyset), \quad \Gamma(W_{-3}, \emptyset) = \Gamma(W_{-2}, \emptyset) + \chi\Gamma(L_{-2}, \emptyset) - \frac{1}{2}w_1, \\ \Gamma(W_{-2}W_{-1}, \emptyset) &= [\Gamma(W_{-2}, \emptyset) + 2\chi]\Gamma(W_{-1}, \emptyset) + \frac{9}{2}D_\infty\Gamma(L_{-1}, \emptyset).\end{aligned}$$

Level [2,1] Similarly, we don't write the explicit forms of  $\Gamma(W_{-1}^2, L_{-1})$  and  $\Gamma(W_{-1}^2, W_{-1})$ .

$$\begin{aligned}\Gamma(L_{-2}, L_{-1}) &= [\Gamma(L_{-2}, \emptyset) - 1]\Gamma(\emptyset, L_{-1}), \quad \Gamma(L_{-2}, W_{-1}) = [\Gamma(L_{-2}, \emptyset) - 1]\Gamma(\emptyset, W_{-1}), \\ \Gamma(L_{-1}^2, L_{-1}) &= \Gamma(L_{-1}, \emptyset)[\Gamma(L_{-1}, L_{-1}) + 2\Delta_0], \quad \Gamma(L_{-1}^2, W_{-1}) = \Gamma(L_{-1}, \emptyset)[\Gamma(L_{-1}, W_{-1}) + 3w_0], \\ \Gamma(L_{-1}W_{-1}, L_{-1}) &= \Gamma(L_{-1}, \emptyset)\Gamma(W_{-1}, L_{-1}) + 2\Delta_0\Gamma(W_{-1}, \emptyset), \\ \Gamma(L_{-1}W_{-1}, W_{-1}) &= \Gamma(L_{-1}, \emptyset)\Gamma(W_{-1}, W_{-1}) + 3w_0\Gamma(W_{-1}, \emptyset) - (\Delta_\infty - \Delta_0)(\Delta_\infty + 1)(\frac{9}{2}D_1 - 2\chi^2), \\ \Gamma(W_{-2}, L_{-1}) &= \Gamma(W_{-2}, \emptyset)\Gamma(\emptyset, L_{-1}) - 2(-w_\infty + w_1 + w_0), \\ \Gamma(W_{-2}, W_{-1}) &= [\Gamma(W_{-2}, \emptyset) - 2\chi]\Gamma(\emptyset, W_{-1}) - \frac{9}{2}D_0\Gamma(\emptyset, L_{-1}),\end{aligned}$$

where  $D_1 := \kappa(\Delta_1 + \frac{1}{5}) - \frac{1}{5}$ .

Level [1,2] Similarly, we don't write the explicit forms of  $\Gamma(L_{-1}, W_{-1}^2)$  and  $\Gamma(W_{-1}, W_{-1}^2)$ .

$$\begin{aligned}\Gamma(L_{-1}, L_{-2}) &= \Gamma(L_{-1}, \emptyset)[\Gamma(\emptyset, L_{-2}) - 1], \quad \Gamma(W_{-1}, L_{-2}) = \Gamma(W_{-1}, \emptyset)[\Gamma(\emptyset, L_{-2}) - 1], \\ \Gamma(L_{-1}, L_{-1}^2) &= [\Gamma(L_{-1}, L_{-1}) + 2\Delta_\infty]\Gamma(\emptyset, L_{-1}), \quad \Gamma(W_{-1}, L_{-1}^2) = [\Gamma(W_{-1}, L_{-1}) + 3w_\infty]\Gamma(\emptyset, L_{-1}), \\ \Gamma(L_{-1}, L_{-1}W_{-1}) &= \Gamma(L_{-1}, W_{-1})\Gamma(\emptyset, L_{-1}) + 2\Delta_\infty\Gamma(\emptyset, W_{-1}), \\ \Gamma(W_{-1}, L_{-1}W_{-1}) &= \Gamma(W_{-1}, W_{-1})\Gamma(\emptyset, L_{-1}) + 3w_\infty\Gamma(\emptyset, W_{-1}), \\ \Gamma(L_{-1}, W_{-2}) &= \Gamma(L_{-1}, \emptyset)\Gamma(\emptyset, W_{-2}) + 2(-w_\infty + w_1 + w_0), \\ \Gamma(W_{-1}, W_{-2}) &= \Gamma(W_{-1}, \emptyset)[\Gamma(\emptyset, W_{-2}) + 2\chi] + \frac{9}{2}D_\infty[\Gamma(\emptyset, L_{-1}) - 2\Delta_1] - (\frac{9}{2}D_1 - 2\chi^2)(-3\Delta_\infty + \Delta_1 + 3\Delta_0).\end{aligned}$$

Level [0,3] Similarly, we don't write the explicit form of  $\Gamma(\emptyset, W_{-1}^3)$ .

$$\begin{aligned}\Gamma(\emptyset, L_{-3}) &= \Gamma(\emptyset, L_{-2}) + \Delta_1, \quad \Gamma(\emptyset, L_{-2}L_{-1}) = [\Gamma(\emptyset, L_{-2}) + 1]\Gamma(\emptyset, L_{-1}), \\ \Gamma(\emptyset, L_{-1}^3) &= [\Gamma(\emptyset, L_{-1}) + 2]\Gamma(\emptyset, L_{-1}^2), \quad \Gamma(\emptyset, L_{-2}W_{-1}) = [\Gamma(\emptyset, L_{-2}) + 1]\Gamma(\emptyset, W_{-1}), \\ \Gamma(\emptyset, L_{-1}^2W_{-1}) &= [\Gamma(\emptyset, L_{-1}) + 2]\Gamma(\emptyset, L_{-1}W_{-1}), \quad \Gamma(\emptyset, L_{-1}W_{-2}) = [\Gamma(\emptyset, L_{-1}) + 2]\Gamma(\emptyset, W_{-2}), \\ \Gamma(\emptyset, L_{-1}W_{-1}^2) &= [\Gamma(\emptyset, L_{-1}) + 2]\Gamma(\emptyset, W_{-1}^2), \quad \Gamma(\emptyset, W_{-3}) = \Gamma(\emptyset, W_{-2}) - \chi\Gamma(\emptyset, L_{-2}) + \frac{1}{2}w_1, \\ \Gamma(\emptyset, W_{-2}W_{-1}) &= [\Gamma(\emptyset, W_{-2}) - 2\chi]\Gamma(\emptyset, W_{-1}) + \frac{9}{2}D_0\Gamma(\emptyset, L_{-1}).\end{aligned}$$

## D Constraint on 3-point function which contains two degenerate operators

In this appendix, we study some properties of 3-point functions  $\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} | \Delta_1, w_1 \rangle$  where two operators have level-1 singular vectors. It may help us to remove the difficulty for general quiver in §4.2 where the 3-point functions (24)–(25) vanishes automatically if the two of the vertex operators have the level-1 null state. We show here that the conformal Ward identity implies much weaker condition for the third vertex operator to have nonvanishing 3-point function. It suggests that we need to modify these formulae for such special case.

We assume the bra and ket state to have such singular vectors:

$$(W_{-1} - \frac{3w_1}{2\Delta_1} L_{-1}) |\Delta_1, w_1\rangle = 0, \quad \langle \Delta_2, w_2 | (W_1 - \frac{3w_2}{2\Delta_2} L_1) = 0. \quad (79)$$

What we study in the following is the consequence of Ward identity explained in eq. (3). In order to make the explanation clearer, it will be useful to introduce some notation in [54].

We start from the action of  $L_n, W_n$  on primary field

$$\begin{aligned} [L_n, \phi_{\Delta, w}(z)] &= z^{n+1} \partial \phi_{\Delta, w} + \Delta(n+1) z^n \phi_{\Delta, w}(z), \\ [W_n, \phi_{\Delta, w}(z)] &= z^n \left\{ \frac{w}{2} (n+1)(n+2) + (n+2) z \hat{W}_{-1} + z^2 \hat{W}_{-2} \right\} \phi_{\Delta, w}(z). \end{aligned} \quad (80)$$

We define the operators as

$$\begin{aligned} e_n(z) &:= L_n - 2zL_{n-1} + z^2L_{n-2} \\ f_n(z) &:= W_n - 3zW_{n-1} + 3z^2W_{n-2} - z^3W_{n-3}, \end{aligned} \quad (81)$$

which satisfy

$$[e_n(z), \phi_{\Delta, w}(z)] = [f_n(z), \phi_{\Delta, w}(z)] = 0. \quad (82)$$

By combining them with highest weight condition, we find

$$\begin{aligned} \langle \Delta_2, w_2 | \phi_{\Delta_3, w_3}(z) e_n(z) &= \langle \Delta_2, w_2 | \phi_{\Delta_3, w_3}(z) f_n(z) = 0 \quad (\text{for } n < 0), \\ \langle \Delta_2, w_2 | \phi_{\Delta_3, w_3}(z) (e_0(z) - \Delta_2) &= \langle \Delta_2, w_2 | \phi_{\Delta_3, w_3}(z) (f_0(z) - w_2) = 0. \end{aligned} \quad (83)$$

Suppose the bra state  $\langle \Delta_2, w_2 |$  has also the level-1 null state as eq. (79), then it gives a constraint on the operator  $\phi_{\Delta_3, w_3}$  to have a nonvanishing

3-point function. To derive it, we rewrite the level-1 null state condition for  $\phi_{\Delta_2, w_2}$  in eq. (79) as

$$\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} (f_1 - \frac{3w_2}{2\Delta_2} e_1) = 0, \quad (84)$$

In the following we put  $z = 1$  and omit the argument of field in the following.

The action of  $f_1 - \frac{3w_2}{2\Delta_2} e_1$  on the ket vector  $|\Delta_1, w_1\rangle$  is evaluated by

$$(f_1 - \frac{3w_2}{2\Delta_2} e_1) |\Delta_1, w_1\rangle = \left\{ -3(w_1 - \frac{w_2\Delta_1}{\Delta_2}) + \frac{3}{2}(\frac{3w_1}{\Delta_1} - \frac{w_2}{\Delta_2}) L_{-1} - W_{-2} \right\} |\Delta_1, w_1\rangle. \quad (85)$$

The 3-point functions for the descendants are given as

$$\frac{\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} L_{-1} | \Delta_1, w_1 \rangle}{\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} | \Delta_1, w_1 \rangle} = \Delta_1 - \Delta_2 + \Delta_3, \quad (86)$$

$$\frac{\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} W_{-2} | \Delta_1, w_1 \rangle}{\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} | \Delta_1, w_1 \rangle} = 2w_1 + w_2 - w_3 - 3w_1 \frac{\Delta_2 - \Delta_3}{\Delta_1}. \quad (87)$$

By the requirement  $\langle \Delta_2, w_2 | \phi_{\Delta_3, w_3} (f_1 - \frac{3w_2}{2\Delta_2} e_1) | \Delta_1, w_2 \rangle = 0$ , we obtain the constraint on the third vertex

$$w_3 = \frac{3}{2}(\Delta_1 + \Delta_2 - \Delta_3) \left( \frac{w_1}{\Delta_1} - \frac{w_2}{\Delta_2} \right) - w_1 + w_2. \quad (88)$$

As we noted in §4.2, the 3-point coefficients in eqs. (24)–(25) vanish when two of the vertex operators to have level-1 null state. So the relation which we obtained here is much weaker. Conformal symmetry requires only one linear relation between  $\Delta_3$  and  $w_3$  to have nonvanishing 3-point function.

It may be interesting to learn the implication of the formula (88). One possibility may be the third vertex is forced to have level-1 null state as the other two. Since such conclusion would have a serious consequence in AGT-W relation, let us work more on it.

In order to have level-1 singular vector, we need a constraint on  $h, w$ . In order to see it, we need to impose

$$L_1(W_{-1} - \frac{3w}{2\Delta} L_{-1}) |\Delta, w\rangle = 0, \quad W_1(W_{-1} - \frac{3w}{2\Delta} L_{-1}) |\Delta, w\rangle = 0. \quad (89)$$

After using the commutation relations

$$\begin{aligned} [L_1, L_{-1}] &= 2L_0, & [L_1, W_{-1}] &= [W_1, L_{-1}] = 3W_0, \\ \frac{2}{9}[W_1, W_{-1}] &= \kappa\Lambda_0 - \frac{1}{5}L_0, \end{aligned} \quad (90)$$

where  $\kappa = \frac{32}{22+5c}$  and  $\Lambda_0 = L_0^2 + \frac{1}{5}L_0 + \dots$ , we find that the first equation in eq. (89) is satisfied automatically but the second equation requires

$$\left(\frac{w}{\Delta}\right)^2 = \kappa(\Delta + \frac{1}{5}) - \frac{1}{5}. \quad (91)$$

For  $c = 2$  (namely  $Q = 0$ ),  $\kappa$  becomes 1. So this constraint reduces to

$$w^2 = \Delta^3. \quad (92)$$

For this case, we may parametrize

$$\Delta_a = p_a^2, \quad w_a = -p_a^3 \quad (a = 1, 2) \quad (93)$$

and would like to see if similar condition exists for  $\Delta_3, w_3$ . Putting this in eq. (88) with  $\Delta_3 = p_3^2$ , we obtain

$$w_3 = \frac{1}{2}(p_1 - p_2)(3p_3^2 - (p_1 - p_2)^2). \quad (94)$$

Obviously it does not take the form of level-1 null state (which should be equal to  $-p_3^3$ ). However, if we additionally put  $p_3 = p_2 - p_1$ , which may be regarded as “momentum conservation,” we have

$$w_3 = -(p_2 - p_1)^3. \quad (95)$$

Such momentum conservation is needed for free field theory. For such case, the third vertex operator needs to have the level-1 null state. Otherwise, for Toda theory with exponential interaction, the momentum conservation condition is in general broken and the third vertex operator need not to have the level-1 null state.

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